Approche multiéchelle pour le comportement vibratoire des structures avec un défaut de rigidité

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THÈSE

pour obtenir le titre de

Docteur en Sciences
de l'UNIVERSITÉ de Tunis El Manar et de l'UNIVERSITÉ de Nice-Sophia Antipolis

Spécialité

Mathématiques Appliquées

présentée et soutenue par

Nadia BEN BRAHIM ep.BEN YOUSSEF

Titre

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Soutenue le 13 juin 2014

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Remerciements

Je tiens à remercier mes encadreurs M. Hedi Hassis et M. Bernard Rousselet pour leurs encadrements et pour leurs précieux conseils qui sont pour beaucoup dans la réalisation de la présente thèse. Particulièrement Mr.Bernard Rousselet pour m'avoir aidé à obtenir des financements pour pouvoir assister à des colloques ainsi pour ses encouragements et son soutien dans les moments les plus difficiles.

Ma gratitude s'adresse aussi à toute l'équipe des deux laboratoires pour m'avoir fourni les meilleurs conditions du travail ainsi qu'à l'équipe du laboratoire Lamsin de l'école nationale des ingénieurs de Tunis qui m'a adopté à bras ouvert et qui était d'une grande aide moralement et scientifiquement, en particulier, Amine Abdelmoula, Amel Ben Arfa et Moncef Mahjoub.

Je remercie aussi les membres du jury et particulièrement les rapporteurs d'avoir accepté d'évaluer ce travail.

Je suis également reconnaissante à toutes les personnes qui ont participé directement ou indirectement à la réalisation de ce projet, en particulier ma famille.

Et finalement, Je dédie cette thèse à mon papa qui n'est plus avec nous et qui a attendu ce jour arrivé, hélas, il n'est plus là.....

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INTRODUCTION

Des études expérimentales ont mis en évidence la possibilité de détecter des comportements non linéaires dans une structure mécanique en analysant sa réponse vibro-acoustique à une sollicitation extérieure. La réponse fréquentielle à une sollicitation harmonique met en évidence la génération d'harmoniques supérieures et de l'inter-modulation en réponse à une sollicitation à 2 fréquences ; voir par exemple [EDK99], [MCG02]. Une méthode vibro-acoustique utilisant l'inter-modulation a été développée pour détecter des défauts dans des câbles précontraint d'un pont à haubans ; cette méthode a permis de retrouver un câble endommagé détecté par ailleurs par une méthode conventionnelle de comparaison de fréquence fondamentale avec celle mesurée 15 ans plus tôt ; mais cette nouvelle méthode a aussi permis de détecter un autre câble endommagé à une extrémité, [VanLG03],[LVAN04], [RVAN05]. Dans ce travail, nous étudions les vibrations de structures mécaniques à un nombre fini de degrés de liberté ; leur comportement peut être modélisé par un système différentiel en utilisant directement les lois de la mécanique ; c'est une approche habituelle en génie mécanique ; on peut aussi obtenir un tel système en discrétisant par éléments finis un système d'équations aux dérivées partielles de la mécanique des milieux continus ; c'est une approche courante en mécanique théorique et en mathématiques. L'aspect de propagation acoustique n'est pas considéré dans ce travail.

Dans le cas de comportements linéaires, de tels systèmes différentiels peuvent être résolus explicitement à partir des valeurs propres de la matrice de rigidité (ou les valeurs propres généralisées du couple de matrices (K, M) dans le cas d'un système provenant des éléments finis).

Les fréquences de vibrations peuvent être déterminées numériquement par un calcul de valeurs propres pour lesquels de nombreux algorithmes existent.

Il en va tout autrement dans un cas non linéaire ; l'exemple classique du pendule simple

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0$$

met en évidence cette difficulté pour un système à un degré de liberté. On ne peut calculer qu'une approximation de la période du mouvement par exemple avec une intégrale elliptique.

Pour chercher une solution approchée, on peut utiliser une intégration numérique pas à pas comme par exemple une méthode de Runge et Kutta mais la détermination très précise de la fréquence ou le calcul de la solution dans un grand intervalle de temps n'est pas facile. Dans le cas d'une petite solution, on peut envisager d'utiliser une méthode de perturbation.

Pour évoquer une méthode de perturbation, proposons nous de chercher une petite solution pour le pendule sous la forme :

$$\theta = \epsilon\theta_1 + \epsilon^2 r; \tag{1}$$

le point crucial est le fait que r soit borné ; si ce n'est pas le cas, on ne peut pas dire que $\epsilon\theta_1$ est une approximation de θ. Nous commençons par approcher l'équation par

$$\ddot{\theta} + \omega_0^2\left(\theta - \frac{\theta^3}{6}\right) = 0;$$

en identifiant les puissances de ϵ, on obtient

$$\ddot{\theta}_1 + \omega_0^2\theta_1 = 0 \tag{2}$$
$$\ddot{r} + \omega_0^2\left(r - \epsilon\theta_1^3 + \epsilon^2\phi(\epsilon, \theta_1, r)\right) = 0 \tag{3}$$

Si nous supposons $\dot{\theta}(0) = 0$, la solution de la première équation est $\theta_1 = acos(\omega_0 t)$ et l'on a $\theta_1^3 = a^3\dfrac{cos(3\omega_0 t) + 3cos(\omega_0 t)}{4}$; dans ces conditions, la solution de la deuxième équation contient un terme du type $ctsin(\omega_0 t)$;

ce terme n'est pas borné sur $[0, +\infty[$ et le développement (1) n'est donc pas satisfaisant sur un intervalle du type $[0, 1/\epsilon[$ mais seulement sur un intervalle borné indépendant de ϵ.

D'autre part, on peut vérifier que $cos((\omega_0+\epsilon)t)-cos(\omega_0 t)$ ne peut pas être majoré uniformément en t par $c\epsilon$; en effet $\lim_{\epsilon\to 0} \frac{cos((\omega_0+\epsilon)t)-cos(\omega_0 t)}{\epsilon} = tsin((\omega_0+\epsilon)t$.

Cette dernière remarque incite à chercher une approximation au premier ordre de la solution sous la forme d'une fonction trigonométrique dont la fréquence dépend de ϵ. Ceci peut être réalisé de diverses manières. Citons la méthode de Lindstedt-Poincaré bien adaptée pour les solutions périodiques ; elle consiste à faire un changement de variable temps dans l'équation différentielle sous la forme $s = \omega_\epsilon t$ et à utiliser une méthode de perturbation sur le système transformé ; on détermine en même temps une approximation de la solution et de sa fréquence. Cette méthode ancienne a été utilisée récemment pour trouver un développement pour un système différentiel qui modélise des vibrations avec non linéarités de contact, [JR10], [VLP08].

Toutefois dans le cas d'un système mécanique amorti et forcé on doit chercher des solutions non nécessairement périodiques ; dans ce cas une méthode à échelle multiples s'est révélée bien adaptée.

Une approche multi échelles, des matériaux à comportement non linéaire sera le contenu de cette étude. Cette méthode est utilisée pour l'approximation en temps d' équations faiblement non linéaires, sachant que les équations de mouvement de la structure sont préalablement discrétisées en espace par une méthode de Galerkin. L'avantage de cette méthode par rapport aux autres méthodes numériques, est que la solution obtenue est analytique et non pas numérique. Elle est donc, moins couteuse en terme de temps de calcul. Aussi, elle trouve directement des solutions où leurs comportements asymptotiques sont des fonctions périodiques.

Le cas de contact rigide qui est également important du point de vue de la théorie et les applications a été abordé dans plusieurs articles, par exemple [JL01], et une synthèse [BBL13], une méthode numérique pour calculer des solutions périodiques est proposée dans[LL11]. Dans [JPS04] une approche numérique pour les grandes solutions de systèmes linéaires par morceaux est proposé. Un document de synthèses sur "modes normaux non linéaires" peut être trouvée dans [KPGV09], il comprend de nombreux articles publiés par la communauté du génie mécanique ; plusieurs domaines d'application ont été abordés par cette communauté, par exemple dans [Mik10] "nonlinear vibro-absorption problem, the cylindrical shell nonlinear dynamics and the vehicle suspension nonlinear dynamics are analyzed".

Des versions préliminaires de ces résultats peuvent être trouvées dans [BR09] et ont été présentées à des conférences [Bra10, Bra] ; une preuve de convergence du développement double échelle est en [BRa13].

Dans un prochain article, le cas non lisse sera considéré ainsi qu'un algorithme numérique basée sur la méthode du point fixe utilisée dans [Rou11]. Ces systèmes vibrants reliés à une barre génèrent des ondes acoustiques ; ce point sera étudié dans un autre travail ; voir Junca-Lombard [JL09]. C.Touzé et M.Amabili ont construit des modèles d'ordre réduit pour des structures harmoniques avec une non linéarité géométrique en utilisant le principe des modes normaux non linéaire définit par Rosenberg [ROS66] alors que G.Kerschen et al. [KPGV09] ont introduit la notion de modes non linéaires et cela par une analyse temps -fréquence permettant la détermination des caractéristiques.

Historiquement, les développements asymptotiques ont été utilisés depuis la célèbre mémoire de Poincaré [Poi99]. Dans le cadre du génie mécanique, A. Nayfeh et des coauteurs ont utilisé ces développements pour trouver la solution de nombreux exemples ainsi qu'étudier le cas de résonance primaire ou secondaire. Il a présenté une comparaison entre la méthode des développements asymptotiques à triple échelle et la méthode de la moyenne en proposant des techniques afin de résoudre une incohérence entre la solution approchée obtenue par la première méthode et celle de la moyenne [Nay05]. Murdock a analysé plusieures méthodes, y compris les développements à échelles multiples dans le livre [Mur91]. Donald R. Smith a présenté dans un livre l'approche par multiples échelles comme une technique pour étudier les équations aux dérivées partielles qui modélisent les propriétés macroscopiques des milieux composites et pour déterminer la solution approchée pour les problèmes d'oscillations [Sm85]. De nombreux autres articles pourraient être cités.

S. Junca et H. Hazim membres du laboratoire J.A.Dieudonné à l'université de Nice Sophia Antipolis, ont introduit la méthode des développements asymptotiques de Lindstedt-Poincaré dans le cadre de la détermination de la réponse vibratoire d'un système avec une non linéarité de type contact unilatéral avec des ressorts [JR10, HR09a].

Cependant un des objectifs de cette thèse, est de caractériser les fréquences de résonance qui apparaissent lors de l'étude de la réponse, ces approches ont été utilisées pour déterminer une solution approchée des systèmes en vibrations avec une non linéarité localisée de rigidité. Les lois de comportement ont été approchées par un polynôme de troisième degré avec un terme quadratique et cubique. Le problème obtenu a été discrétisé spatialement par éléments finis conduisant ainsi à un système d'équations différentielles non linéaires du deuxième ordre. Le système discret a été ensuite projeté dans la base des vecteurs propres où la réponse approchée d'un système à n degrés de libertés a été déterminée en utilisant les développements asymptotiques à multiples échelles et le principe des modes normaux qui sont présentés comme une extension naturelle des modes normaux linéaires ; le comportement de la solution est alors très proche du cas à un degré de liberté. Auparavant le cas à un degré de liberté a été entièrement résolu pour mettre en place la méthode.

Le calcul de ces modes non linéaires donne accès à une meilleure compréhension de l'effet de la non linéarité sur ses modes propres linéaires conduisant à la compréhension du comportement dynamique du système mécanique analysé. Un mode normal non linéaire est défini comme une solution périodique où toutes les composantes du système mécanique vibrent à la même fréquence ; cette solution peut être trouvée en excitant le système par une force particulière ayant une fréquence proche de celle d'un mode propre non linéaire. Les fréquences de résonances, qui apparaissent lors de l'étude de la réponse vibratoire forcée, sont caractérisées.

Dans un premier chapitre, cette approche est explicitée à travers un développement asymptotique double échelle d'un système à un degré de liberté puis à plusieurs degrés de liberté. C'est l'objet de l'article ≪ Double scale analysis of periodic solutions of some non linear vibrating systems ≫. La non linéarité de la loi de comportement contrainte-déformation contient un terme quadratique ainsi qu'un terme cubique. On cherche une petite solution $\tilde{u} = \epsilon u$ où ϵ est un petit paramètre. Le cas de vibration libre est considéré en premier lieu, tout en présentant une justification de ces développements. Ces approches permettent d'obtenir une solution périodique proche d'un mode normal linéaire (appelé mode normal non linéaire). Les résultats obtenus par ces approches ont été comparés avec une résolution numérique pas à pas utilisant le logiciel Scilab ; une bonne concordance a été remarquée pour ϵ très petit ; ce développement obtenu n'est valable que pour de petites solutions et de basses fréquences. Ensuite, une transformation de Fourier numérique a été effectuée pour déterminer les fréquences. Celles ci ont été comparées avec celles du système linéaire, les solutions peuvent tout au moins servir de références pour identifier les effets de la non linéarité localisée sur le comportement non linéaire global.

Dans une deuxième partie de ce premier chapitre, nous arrivons à décrire un phénomène de résonance qui étend la résonance classique du cas linéaire. Pour le système forcé la présence d'un terme d'amortissement est essentielle pour obtenir une solution convenable ; la fréquence d'excitation est considérée proche de celle du système linéaire, une étude d'un tel système a été faite et l'approche par développement asymptotique double échelle avec un paramètre perturbateur ϵ a été appliquée tout en incluant une preuve rigoureuse de la convergence de la méthode de double échelles. Auparavant, une étude de la solution stationnaire ainsi que la stabilité de la solution dynamique au voisinage de la solution stationnaire a été faite pour prouver la convergence. Cette étude a fourni une solution approchée périodique, par suite, une approximation de la fréquence dont l'amplitude atteint une valeur maximale (*résonance primaire*). Nous remarquerons dans l'expression de la fréquence angulaire amortie, qu'aux termes trouvés dans le cas de vibration libre s'ajoute un terme impliquant le facteur d'amortissement. Pour cette fréquence, une solution approchée périodique a été obtenue à l'ordre ϵ. L'intéret de ce résultat est que lorsque la solution stationnaire atteint son amplitude maximum, la force excitatrice satisfait la relation suivante $F = \lambda\omega a$ comme dans le cas linéaire ; ainsi, le terme d'amortissement λ peut être déterminé ; c'est assez intéressant en pratique car il est en général difficile à mesurer. Par contre, la non linéarité quadratique est absente dans les expressions de la solution et de la fréquence, elle n'intervient que dans la justification du développement d'où l'idée passer à l'utilisation de trois échelles.

Le deuxième chapitre est constitué d'un deuxième article qui s'intitule ≪ Triple scale analysis of periodic solutions and resonance of some asymmetric non linear vibrating systems ≫ ; il est publié dans [BR13].

La démarche du cas double échelle a été suivie pour l'étude de l'analyse avec trois échelles de la réponse vibratoire pour un système à un degré de liberté puis pour n degrés de liberté en vibration libre puis amortie et forcée. Malgré la ressemblance de la démarche, beaucoup de différences ont été notées. Nous considérons d'abord une réponse libre. En ajoutant une troisième échelle de temps et en développant à l'ordre deux, l'expression de la solution approchée a changé. Aux termes trouvés avec une analyse à deux échelle de temps s'ajoute un terme d'ordre ϵ^2 qui fait intervenir les termes quadratiques et cubiques dans l'expression de la fréquence. Il a été observé que le coefficient du terme quadratique de la loi de comportement n'a aucune influence sur la fréquence de la réponse approchée obtenue avec le développement à double échelle mais ce coefficient intervient dans la fréquence obtenue avec le développement avec trois échelles.

Pour la deuxième partie de ce deuxième chapitre, pour la réponse forcée amortie, la méthode de perturbation triple échelle a été appliquée. Afin de justifier le développement, l'étude de la solution stationnaire ainsi que la stabilité de la solution au voisinage de cette solution stationnaire a été faite tout en précisant l'approximation de la solution. L'introduction d'une autre échelle de temps a compliqué certes la détermination de l'amplitude et de la phase. Cette difficulté était connue dans la comunauté de génie mécanique ; nous avons eu recourt à la méthode de la reconstitution proposée dans [Nay05] qui semble satisfaire la problématique mécanique ; nous avons donné une preuve mathématique de l'approximation de la solution avec cette approche. Un lien entre la fréquence de la solution libre et l'amplitude de la solution forcée a été mis en évidence. L'étude du maximum de la solution stationnaire nous a permis de déterminer la fréquence de résonance du système forcé. Des versions préliminaires de ces résultats peuvent être trouvés dans [BR09] et ont été présentés à $\ll$ SMAI conférence de 2009 $\gg$et au $\ll$ GDR-AFPAC 2010 Conférence $\gg$.

Pour le comportement mécanique, des développements quadratiques et cubiques ont été ici proposés, en utilisant un développement à double puis à triple échelles. Cette association conduit à un calcul lourd mais bien justifié mathématiquement et pour ϵ assez petit avec un bon accord avec une solution numérique ; il est limité à une loi de comportement élastique.

Dans un troisième chapitre, on considère une loi de comportement élastoplastique écrouissable, un modèle unifié a été présenté au chapitre (3) ; ce modèle est considéré comme une perspective possible de cette thèse. Des exemples de validation ont été testés pour montrer l'efficacité de ce modèle.

Chapitre 1

Double scale analysis of periodic solutions of some non linear vibrating systems

Ce travail fait l'objet d'un article, déposé dans l'archive ouverte pluridisciplinaire H.A.L[1] du Centre pour la communication scientique directe

[1] *hal.archives-ouvertes.fr/hal-00776184*

Double scale analysis of periodic solutions of some non linear vibrating systems

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june 13, 2013

Abstract

We consider *small solutions* of a vibrating system with smooth non-linearities for which we provide an approximate solution by using a double scale analysis; a rigorous proof of convergence of a double scale expansion is included; for the forced response, a stability result is needed in order to prove convergence in a neighbourhood of a primary resonance. **Keywords:** double scale analysis; periodic solutions; nonlinear vibrations, resonance MSC: 34e13, 34c25, 74h10, 74h45

1 Introduction

In this work we look for an asymptotic expansion of *small periodic solutions* of free vibrations of a discrete structure without damping and with local non linearity; then the same system with light damping and a *periodic forcing* with frequency close to a frequency of the free system is analyzed (primary resonance). For a small solution, we recover a behavior with some similarity with the linear case; in particular the amplitude of the forced response reaches a local maximum at the frequency of the free response. On the other hand the frequency of the free response is amplitude dependent and the superposition principle does not apply. The work of Lyapunov [oL49] is often cited as a basis for the existence of periodic solutions which tends towards linear normal modes as amplitudes tend to zero; the proof of this paper uses the *hypothesis of analycity* of the non linearity involved in the differential system. In [Rou11], we addressed the case of a non linearity which is only lipschitzian and we prove existence of periodic solutions with a constructive proof; in this case the result of Lyapunov obviously may not be applied. Non-linearity of oscillations is a classical theme in theoretical physics, for example at master level, see [LL58] in Russian or its English or French translation in [LL60, LL66].

Asymptotic expansions have been used for a long time; such methods are introduced in the famous memoir of Poincaré [Poi99]; a general book on asymptotic methods is [BM55] with french and English translations [BM62, BM61]; introductory material is in [Nay81], [Mil06]; a detailed account of the averaging method with precise proofs of convergence may be found in [SV85]; an analysis of several methods including multiple scale expansion may be found in [Mur91]; the case of vibrations with unilateral springs have been presented in [JR09, JR10, VLP08], [HR09a, HR09b, HFR09, Haz, Haz10]; in [JPS04] a numerical approach for large solutions of piecewise linear systems is proposed. The case of rigid contact which is also important from the point of view of theory and applications has been addressed in several papers, for example [JL01], and a synthesis in [JBL13] . A review paper for so called "non linear normal modes" may be found in [KPGV09]; it includes numerous papers published by the mechanical community; several application fields have been addressed by the mechanical community; for example in [Mik10] "nonlinear vibro-absorption problem, the cylindrical shell nonlinear dynamics and the vehicle suspension nonlinear dynamics are analyzed".

In the mechanical engineering community the validity of the expansions is assumed to hold; however, this is not straightforward as this kind of expansion is not a standard series expansion and the expansion is usually not valid for all time; for example, this point has been raised in [Rub78]. If the averaging method was carefully analyzed as indicated above, it seems not to be the case for the multiple scale method, the expansion of which is often compared to the one obtained by the averaging method.

Here in a first stage we consider *small solutions* of a system with smooth non-linearities for which we provide an approximate solution by using a double scale analysis; a rigorous proof of convergence of the method of double scale is included; for the forced response, a stability result is needed in order to prove convergence. As an introduction, the next section addresses the one degree of freedom case while the following one considers many degrees of freedom; for free vibrations we find solutions close to a linear normal mode (so called non linear normal modes) and for forced vibrations, we describe the response for forcing frequency close to a free vibration frequency. Preliminary versions of these results may be found in [BR09] and have been presented in conferences [Bra10, Bra]; related results have been presented in [Gas]. Triple scale expansions is to be submitted [BR13]. In a forthcoming paper, the non-smooth case will be considered as well as a numerical algorithm based on the fixed point method used in [Rou11].

2 One degree of freedom, strong cubic non linearity

In this section, we consider the case of a mass attached to a spring; in the case of a stress-strain law of the form $n = ku + mcu^2 + mu^3$, we find no shift of frequency at first order, so here we concentrate on a stress-strain law with a stronger cubic non linearity:

$$n = ku + mcu^2 + m\frac{d}{\epsilon}u^3$$

where ϵ is a small parameter which is also involved in the size of the solution as in previous paragraph; the choice of this scaling provides frequencies which are amplitude dependent.

2.1 Free vibration, double scale expansion up to first order

Using second Newton law, free vibrations of a mass attached to such a spring are governed by:

$$\ddot{u} + \omega^2 u + cu^2 + \frac{du^3}{\epsilon} = 0. \tag{1}$$

We look for a *small* solution with a double scale for time; we set

$$T_0 = \omega t, \quad T_1 = \epsilon t, \tag{2}$$

so with $D_0 u = \frac{\partial u}{\partial T_0}, \quad D_1 u = \frac{\partial u}{\partial T_1}$, we obtain

$$\frac{du}{dt} = \omega D_0 u + \epsilon D_1 u, \ \frac{d^2u}{dt^2} = \omega^2 D_0^2 u + 2\epsilon\omega D_0 D_1 u + \epsilon^2 D_1^2 u \tag{3}$$

and we look for a small solution with initial data
$u(0) = \epsilon a_0 + o(\epsilon)$ and $\dot{u}(0) = o(\epsilon)$; we use the *ansatz*

$$u = \epsilon u_1(T_0, T_1) + \epsilon^2 r(T_0, T_1, \epsilon); \tag{4}$$

so we have:

$$\frac{du}{dt} = \epsilon[\omega D_0 u_1 + \epsilon D_1 u_1] + \epsilon^2[\omega D_0 r + \epsilon D_1 r] \tag{5}$$

and

$$\frac{d^2u}{dt^2} = \epsilon\omega^2 D_0^2 u_1 + \epsilon^2[2\omega D_0 D_1 u_1 + \omega^2 D_0^2 r] + \epsilon^3[D_1^2 u_1 + \mathcal{D}_2 r] \tag{6}$$

with

$$\mathcal{D}_2 r = \frac{1}{\epsilon}\left(\frac{d^2r}{dt^2} - \omega^2 D_0^2 r\right) = 2\omega D_0 D_1 r + \epsilon D_1^2 r \tag{7}$$

We plug expansions (4),(6) into (1); by identifying the powers of ϵ in the expansion of equation (1), we obtain:

$$\begin{cases} \omega^2(D_0^2 u_1 + u_1) = 0 \\ (D_0^2 r + r) = \frac{S_2}{\omega^2} \quad \text{with} \end{cases} \tag{8}$$

$$S_2 = -\frac{1}{\epsilon^2}\left[c(\epsilon u_1 + \epsilon^2 r)^2 + \frac{d}{\epsilon}(\epsilon u_1 + \epsilon^2 r)^3\right] - 2\omega D_0 D_1 u_1 - \epsilon \mathcal{R}(u_1, r, \epsilon) \tag{9}$$

where

$$\mathcal{R} = D_1^2 u_1 + \mathcal{D}_2 r; \tag{10}$$

we can manipulate to obtain:

$$S_2 = -\left[c u_1^2 + d u_1^3 + 2\omega D_0 D_1 u_1 + \epsilon R(u_1, r, \epsilon)\right] \tag{11}$$

where

$$R(u_1, r, \epsilon) = \left[\mathcal{R} + 2c u_1 r + 3d u_1^2 r + \epsilon \rho(u_1, r, \epsilon)\right] \tag{12}$$

with a polynomial $\rho(u_1, r, \epsilon) = cr^2 + 3du_1 r^2 + \epsilon d r^3$.

We set $\theta(T_0, T_1) = T_0 + \beta(T_1)$ noticing $D_0\theta = 1$, $D_1\theta = D_1\beta$; we solve equation (8) with:

$$u_1 = a(T_1)\cos(\theta) \tag{13}$$

and we obtain

$$S_2 = \frac{-ca^2}{2}(1 + \cos(2\theta)) - \frac{da^3}{4}(\cos(3\theta) + 3\cos(\theta)) + \\ 2\omega(D_1 a \sin(\theta) + a D_1\beta\cos(\theta)) - \epsilon R(u_1, r, \epsilon); \tag{14}$$

we gather terms at angular frequency 1:

$$S_2 = -\frac{da^3}{4} 3\cos(\theta) + 2\omega\left[D_1 a \sin(\theta) + a D_1\beta\cos(\theta)\right] + S_2^\sharp - \epsilon R(u_1, r, \epsilon) \tag{15}$$

where

$$S_2^\sharp = \frac{-ca^2}{2}(1 + \cos(2\theta)) - \frac{da^3}{4}\cos(3\theta). \tag{16}$$

By imposing

$$D_1 a = 0 \text{ and } \quad 2\omega a D_1\beta = 3\frac{da^3}{4}, \text{ so that} \\ a = a_0, \quad \beta = \beta_0 T_1 \text{ with } \beta_0 = 3\frac{da^2}{8\omega}T_1, \tag{17}$$

we get that $S_2 = S_2^\sharp - \epsilon R(u_1, r, \epsilon)$ no longer contains any term at frequency 1.

In order to show that r is bounded, after eliminating terms at angular frequency 1, we go back to the t variable in the second equation (8).

$$\ddot{r} + \omega^2 r = \frac{\tilde{S}_2}{\omega^2} \quad \text{with} \tag{18}$$

$$\tilde{S}_2 = S_2^\sharp(t,\epsilon) - \epsilon \tilde{R}(u_1, r, \epsilon) \text{ where} \tag{19}$$

$$S_2^\sharp(t,\epsilon) = \frac{-ca^2}{2}\left[1 + \cos(2(\omega t + \beta(\epsilon t)))\right] - \frac{da^3}{4}\cos(3(\omega t + \beta(\epsilon t))) \tag{20}$$

$$= \frac{-ca^2}{2}(1 + \cos(2(\omega t + \beta_0 \epsilon t)) - \frac{da^3}{4}(\cos(3(\omega t + \beta_0 \epsilon t))) \tag{21}$$

$$\text{with } \tilde{R}(u_1, r, \epsilon) = R(u_1, r, \epsilon) - \mathcal{D}_2 r \tag{22}$$

in which the remainder $\tilde{R}$ is expressed with variable t.

Proposition 2.1. *There exists $\gamma > 0$ such that for all $t \leq t_\epsilon = \frac{\gamma}{\epsilon}$, the solution of (1), with $u(0) = \epsilon a_0 + o(\epsilon)$, $\dot{u}(0) = o(\epsilon)$, satisfies the following expansion*

$$u(t) = \epsilon a_0 \ \cos(\nu_\epsilon t) \ + \ \epsilon^2 r(\epsilon, t)$$

where

$$\nu_\epsilon = \omega + 3\epsilon \frac{da^2}{8\omega} \tag{23}$$

and r is uniformly bounded in $C^2(0, t_\epsilon)$.

Proof. Let us use lemma 5.1 with equation (18); set $S = S_2^\sharp$; as we have enforced (17), it is a periodic bounded function orthogonal to $e^{\pm it}$, it satisfies lemma hypothesis; similarly set $g = \tilde{R}$; it is a polynomial in variable r with coefficients which are bounded functions, so it is a lipschitzian function on bounded subsets and satisfies lemma hypothesis. $\square$

2.2 Forced vibration, double scale expansion of order 1

2.2.1 Derivation of the expansion

Here we consider a similar system with a sinusoidal forcing at a frequency close to the free frequency (so called primary resonance); in the linear case, without damping, it is well known that the solution is no longer bounded when the forcing frequency goes to the free frequency. Here, we consider the mechanical system of previous section but with periodic forcing and we include some light damping term; the scaling of the forcing term is chosen so that the expansion works properly; this is a known difficulty, for example see [Nay86].

$$\ddot{u} + \omega^2 u + \epsilon\lambda\dot{u} + cu^2 + \frac{du^3}{\epsilon} = \epsilon^2 F \cos(\tilde{\omega}_\epsilon t). \tag{24}$$

We assume positive damping, $\lambda > 0$ and excitation frequency $\tilde{\omega}_\epsilon$ is *close* to an eigenfrequency of the linear system in the following way:

$$\tilde{\omega}_\epsilon = \omega + \epsilon\sigma. \tag{25}$$

We look for a small solution with a double scale expansion; to simplify the computations, the fast scale T_0 is chosen ϵ dependent and we set:

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t \text{ and } \quad D_0 u = \frac{\partial u}{\partial T_0}, \; D_1 u = \frac{\partial u}{\partial T_1}; \tag{26}$$

so

$$\frac{du}{dt} = \tilde{\omega} D_0 u + \epsilon D_1 u \text{ and } \frac{d^2u}{dt^2} = \tilde{\omega}_\epsilon^2 D_0^2 u + 2\epsilon\tilde{\omega}_\epsilon D_0 D_1 u + \epsilon^2 D_1^2 u; \tag{27}$$

equation (25) provides

$$\tilde{\omega}_\epsilon^2 = \omega^2 + 2\epsilon\omega\sigma + \epsilon^2\sigma^2. \tag{28}$$

With (25), (26), (27), (28) and the *ansatz*

$$u = \epsilon u_1(T_0, T_1) + \epsilon^2 r(T_0, T_1, \epsilon), \tag{29}$$

we obtain:

$$\frac{du}{dt} = \epsilon\frac{du_1}{dt} + \epsilon^2\frac{dr}{dt} = \epsilon\frac{du_1}{dt} + \epsilon^2\omega D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r) = \tag{30}$$

$$\epsilon[\tilde{\omega} D_0 u_1 + \epsilon D_1 u_1] + \epsilon^2\omega D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r) = \tag{31}$$

$$\epsilon[\omega D_0 u_1 + \epsilon\sigma D_0 u_1 + \epsilon D_1 u_1] + \epsilon^2\omega D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r) \tag{32}$$

where we remark that $\frac{dr}{dt} - \omega D_0 r = \epsilon\sigma D_0 r + \epsilon D_1 r$ is of degree 1 in ϵ. For the second derivative, as for the case without forcing, we introduce

$$\mathcal{D}_2 r = \frac{1}{\epsilon}\left(\frac{d^2r}{dt^2} - \omega^2 D_0^2 r\right) \text{ with the expansion} \tag{33}$$

$$\mathcal{D}_2 r = 2\omega[\sigma D_0^2 r + D_0 D_1 r] + \epsilon\left[\sigma^2 D_0^2 r + 2\sigma D_0 D_1 r + D_1^2 r\right]; \tag{34}$$

$$\frac{d^2u}{dt^2} = \epsilon\frac{d^2u_1}{dt^2} + \epsilon^2\frac{d^2r}{dt^2} = \epsilon\frac{d^2u_1}{dt^2} + \epsilon^2\omega^2 D_0^2 r + \epsilon^3\mathcal{D}_2 r \tag{35}$$

$$= \epsilon\left[\tilde{\omega}^2 D_0^2 u_1 + 2\epsilon\tilde{\omega} D_0 D_1 u_1 + \epsilon^2 D_1^2 u_1\right] \tag{36}$$

$$+ \epsilon^2\omega^2 D_0^2 r + \epsilon^3\mathcal{D}_2 r \tag{37}$$

$$= \epsilon\big\{\omega^2 D_0^2 u_1 + 2\epsilon\omega\left(\sigma D_0^2 u_1 + D_0 D_1 u_1\right) + \tag{38}$$

$$\epsilon^2\left[\sigma^2 D_0^2 u_1 + 2\sigma D_0 D_1 u_1 + D_1^2 u_1\right]\big\} \tag{39}$$

$$+ \epsilon^2\omega^2 D_0^2 r + \epsilon^3\mathcal{D}_2 r \tag{40}$$

the last term in the right hand side will be part of the remainder R of equation (42). We plug previous expansions into (24); we obtain:

$$\begin{cases} \omega^2(D_0^2 u_1 + u_1) = 0 \\ D_0^2 r + r = \frac{S_2}{\omega^2} \quad \text{with} \end{cases} \tag{41}$$

$$S_2 = -\left\{cu_1^2 + du_1^3 + 2\omega[D_0D_1u_1 + \sigma D_0^2 u_1] + \lambda\omega D_0 u_1\right\} \qquad (42)$$

$$+F\cos(T_0) - \epsilon R(u_1, r, \epsilon) \qquad (43)$$

and with

$$R(u_1, r, \epsilon) = D_1^2 u_1 + 2cu_1 r + 3du_1^2 r + \sigma^2 D_0^2 u_1 + 2\sigma D_0 D_1 u_1 + \qquad (44)$$

$$\lambda(\omega D_0 r + \sigma D_0 u_1 + D_1 u_1) + \mathcal{D}_2 r \qquad (45)$$

$$+\lambda(\frac{dr}{dt} - \omega D_0 r) + \epsilon\rho(u_1, r, \epsilon). \qquad (46)$$

Set $\theta(T_0, T_1) = T_0 + \beta(T_1)$. We solve the first equation of (41) :

$$u_1 = a(T_1)\cos(\theta); \qquad (47)$$

then we use $T_0 = \theta(T_0, T_1) - \beta(T_1)$ and we obtain

$$\begin{aligned} S_2 = &\frac{-ca^2}{2}(1 + \cos(2\theta)) - \frac{da^3}{4}(\cos(3\theta) + 3\cos(\theta)) + \\ &2\omega(D_1 a\sin(\theta) + aD_1\beta\cos(\theta)) + 2\sigma\omega a\cos(\theta) + a\lambda\omega\sin(\theta) \\ &+ F\sin(\theta)\sin(\beta(T_1)) + F\cos(\theta)\cos(\beta(T_1)) - \epsilon R(u_1, r, \epsilon) \end{aligned} \qquad (48)$$

or

$$\begin{aligned} S_2 = &[2\omega D_1 a + \lambda a\omega + F\sin(\beta)]\sin(\theta) \\ &+ \left[2\omega a D_1\beta + 2\sigma\omega a - \frac{3da^3}{4} + F\cos(\beta)\right]\cos(\theta) \\ &+ S_2^\sharp \quad - \epsilon R(u_1, r, \epsilon) \end{aligned} \qquad (49)$$

with

$$S_2^\sharp = \frac{-ca^2}{2}(1 + \cos(2\theta)) - \frac{da^3}{4}(\cos(3\theta)); \qquad (50)$$

note that $S_2^\sharp$ is a periodic function with frequency strictly multiple of 1.

Orientation. By enforcing

$$\left\{ \begin{array}{l} 2\omega D_1 a + \lambda a\omega = -F\sin(\beta) \\ 2a\omega D_1\beta + 2\sigma\omega a - \frac{3da^3}{4} = -F\cos(\beta) \end{array} \right. \qquad (51)$$

$S_2 = S_2^\sharp - \epsilon R(u_1, r, \epsilon)$ contains neither term at frequency 1 nor at a frequency which goes to 1; this point will enable to justify this expansion under some conditions; before, we study stationary solution of this system and the stability of the dynamic solution in a neighborhood of the stationary solution.

2.2.2 Stationary solution and stability

Let us consider the stationary solution of (51), it satisfies:

$$\left\{ \begin{array}{rl} \lambda a\omega & +F\sin(\beta) = 0 \\ \left(2\omega\sigma - \frac{3da^2}{4}\right) & +\frac{F\cos(\beta)}{a} = 0. \end{array} \right. \tag{52}$$

Now, we study the stability of the solution of (51), in a neighborhood of this stationary solution noted $(\bar{a}, \bar{\beta})$; set $a = \bar{a} + \tilde{a}, \beta = \bar{\beta} + \tilde{\beta}$, the linearized system is written

$$\begin{pmatrix} D_1\tilde{a} \\ D_1\tilde{\beta} \end{pmatrix} = J \begin{pmatrix} \tilde{a} \\ \tilde{\beta} \end{pmatrix};$$

manipulating, we obtain the jacobian matrix.

$$J = \begin{pmatrix} -\frac{\lambda}{2} & -\frac{F}{2\omega}\cos(\bar{\beta}) \\ \frac{9d\bar{a}}{8\omega} - \frac{\sigma}{\bar{a}} & \frac{F}{2\omega\bar{a}}\sin(\bar{\beta}) \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{2} & a(\sigma - \frac{3d\bar{a}^2}{8\omega}) \\ \frac{9d\bar{a}}{8\omega} - \frac{\sigma}{\bar{a}} & -\frac{\lambda}{2} \end{pmatrix}. \tag{53}$$

The matrix trace is $-\lambda$, and the determinant is

$$det(J) = \frac{\lambda^2}{4} - (\frac{9d\bar{a}^2}{8\omega} - \sigma)(\sigma - \frac{3d\bar{a}^2}{8\omega};)$$

we notice that the determinant is strictly positive for $\sigma = 0$ so by continuity, it remains positive for σ small; moreover $\frac{d}{d\sigma}det(J) < 0$ for $\sigma < 0$ so $det(J) > 0$ for $\sigma < 0$; by studying the trinomial in σ, we notice that the determinant is positive when this semi-implicit inequality is satisfied: $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$; so in these conditions, the two eigenvalues are negative; then the solution of the linearized system goes to zero; with the theorem of Poincaré-Lyapunov (look in the appendix for the theorem 5.1,) when the initial data is close enough to the stationary solution, the solution of the system (51), goes to the stationary solution. We expand this point, set

$$y = \begin{pmatrix} a \\ \beta \end{pmatrix}, \quad G(y) = \begin{pmatrix} -\lambda a\omega & -F\sin(\beta) \\ -\left(2\omega\sigma - \frac{3da^2}{4}\right) & -\frac{F\cos(\beta)}{a} \end{pmatrix}; \tag{54}$$

the system (52) may be written $\dot{y} = G(y)$; denote $\bar{y} = \binom{\bar{a}}{\bar{\beta}}$ the solution of (52); perform the change of variable $y = \bar{y} + x$, we have $G(\bar{y} + x) = G(\bar{y}) + Jx + g(x)$ with $g(x) = o(\|x\|$; the theorem 5.1 may be applied with $A = J,\ B = 0$, here the function g does not depends on time.

Proposition 2.2. *If* $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$ *, the stationary solution of* (51) *is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary solution of* (52)*, it remains close to it and converges to it); to the stationary case corresponds the approximate solution of* (24) $u_1 = \bar{a}\cos(T_0 + \bar{\beta})$*, it is periodic; for an initial data close enough to this stationary solution,* $u_1 = a(T_1)\cos(T_0 + \beta(T_1))$ *with* a, β *solutions of* (51)*; it goes to the solution* (52) $\bar{a}, \bar{\beta}$ *when* $T_1 \longrightarrow +\infty$*.*

With this result of stability, we can state precisely the approximation of the solution of (24) by the function u_1.

2.2.3 Convergence of the expansion

Proposition 2.3. *Consider the solution of* (24) *with*

$$u(0) = \epsilon a_0 + o(\epsilon),\ \dot{u}(0) = -\epsilon\omega a_0 \sin(\beta_0) + o(\epsilon),$$

with a_0, β_0 *close of the stationary solution* $(\bar{a}, \bar{\beta})$,

$$|a_0 - \bar{a}| \leq \epsilon C_1, |\beta_0 - \bar{\beta}| \leq \epsilon C_2;$$

When $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{3d\bar{a}^2}{2\omega} - \lambda^2}$, *there exists* $\gamma > 0$ *such that for all* $t \leq t_\epsilon = \frac{\gamma}{\epsilon}$, *the following expansion is satisfied*

$$u(t) = \epsilon a(\epsilon t)\ \cos(\tilde{\omega}_\epsilon t + \beta(\epsilon t))\ +\ \epsilon^2 r(\epsilon, t)$$

with $\omega_\epsilon = \omega + \epsilon\sigma$ *and* r *uniformly bounded in* $C^2(0, t_\epsilon)$ *and with* $a,\ \beta$ *solution of* (51).

Proof. Indeed after eliminating terms at frequency 1, we go back to the variable t for the second equation (41)

$$\ddot{r} + \omega^2 r = \frac{\tilde{S}_2}{\omega^2} \text{ with} \tag{55}$$

$$\tilde{S}_2 = S_2^\sharp(t, \epsilon) - \epsilon\tilde{R}(u_1, r, \epsilon) \tag{56}$$

where

$$\tilde{R}(u_1, r, \epsilon) = R(u_1, r, \epsilon) - \mathcal{D}_2 r - \lambda(\frac{dr}{dt} - \omega D_0 r) \tag{57}$$

with all the terms expressed with the variable t; we have

$$S_2^\sharp(t, \epsilon) = \frac{-ca^2(\epsilon t)}{2}(1 + \cos(2(\tilde{\omega}_\epsilon t + \beta(\epsilon t))) - \frac{da^3(\epsilon t)}{4}\,(\cos(3(\tilde{\omega}_\epsilon t + \beta(\epsilon t))\,; \tag{58}$$

this function is not periodic but is *close* of the periodic function:

$$S_2^\natural(t, \epsilon) = \frac{-c\bar{a}^2}{2}(1 + \cos(2(\tilde{\omega}_\epsilon t + \bar{\beta})) - \frac{da^3}{4}\left(\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta})\right) \tag{59}$$

and for $t \leq \frac{\gamma}{\epsilon}$ as the solution of (51) is stable: it remains close to the stationary solution

$$|a(\epsilon t) - \bar{a}| \leq \epsilon C_1, \quad |\beta(\epsilon t) - \bar{\beta}| \leq \epsilon C_2 \tag{60}$$

and

$$|S_2^\sharp - S_2^\natural| \leq \epsilon C_3; \tag{61}$$

so this difference may be included in the remainder $\tilde{R}$. We use lemma 5.1 with $S = S_2^\natural$; it satisfies lemma hypothesis; similarly, we use $g = \tilde{R}$; it satisfies the hypothesis because it is a polynomial in the variables r, u_1, ϵ, with coefficients which are bounded functions, so it is lipschitzian on bounded subsets. ∎

2.2.4 Maximum of the stationary solution, primary resonance

Consider the stationary solution of (51), it satisfies

$$\left\{ \begin{array}{rl} \lambda a\omega & = -F\sin(\beta) \\ a\left(2\omega\sigma - \frac{3da^2}{4}\right) & = -F\cos(\beta) \end{array} \right. \tag{62}$$

manipulating, we get that a is solution of the equation:

$$f(a,\sigma) = \lambda^2 a^2 \omega^2 + a^2 \left(2\omega\sigma - \frac{3da^2}{4}\right)^2 - F^2 = 0. \tag{63}$$

We compute

$$\frac{\partial f}{\partial \sigma} = 4a^2\omega(2\omega\sigma - \frac{3da^2}{4}) \tag{64}$$

$$\frac{\partial f}{\partial a} = 2a\lambda^2\omega^2 + 2a\left(2\omega\sigma - \frac{3da^2}{4}\right)^2 - 6\frac{da^3}{4}\left(2\omega\sigma - \frac{3da^2}{4}\right) \tag{65}$$

$$\frac{\partial^2 f}{\partial \sigma^2} = 8a^2\omega^2 \tag{66}$$

$$\tag{67}$$

For σ close enough to the solution of $\frac{\partial f}{\partial \sigma} = 0$, $\frac{\partial f}{\partial \sigma}$ is small, $\frac{\partial f}{\partial a}$ is not zero, and with the implicit function theorem this equation defines a function $a(\sigma)$; lets use :

$$\frac{\partial a}{\partial \sigma} = -\frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial f}{\partial a}} \quad \text{and} \quad \frac{\partial^2 a}{\partial \sigma^2} = -\frac{\frac{\partial^2 f}{\partial \sigma^2}}{\frac{\partial f}{\partial a}}.$$

In our case, when
$\frac{\partial a}{\partial \sigma} = 0$, we have

$$\sigma = \frac{3da^2}{8\omega}, \quad \frac{\partial f}{\partial a} = 2a\lambda^2\omega^2, \quad \frac{\partial^2 f}{\partial \sigma^2} = 8a^2\omega^4, \tag{68}$$

so the second derivative $\frac{\partial^2 a}{\partial \sigma^2} < 0$ and a is maximum at the frequency of the free periodic solution.

Proposition 2.4. *The stationary solution of* (51) *satisfies*

$$\left\{ \begin{array}{rl} \lambda a\omega & +F\sin(\beta) = 0 \\ 2a\omega\sigma - \frac{3da^3}{4} & +F\cos(\beta) = 0 \end{array} \right. \tag{69}$$

it reaches its maximum amplitude for $\sigma = \frac{3da^2}{8\omega}$ and $\beta = \frac{\pi}{2} + k\pi$; the excitation is at the angular frequency

$$\tilde{\omega}_\epsilon = \omega + 3\epsilon\frac{da^2}{8\omega} + O(\epsilon^2) \quad \textit{and} \quad F = \lambda\omega a$$

it is the angular frequency ν_ϵ of the free periodic solution (23) *for this frequency, the approximation (of the solution up to the order ϵ) is periodic:*

$$u(t) = \epsilon \frac{F}{\lambda\omega} \sin(\tilde{\omega}_\epsilon t) + \epsilon^2 r(\epsilon, t) \tag{70}$$

Remark 2.1. *We remark that this value of $\sigma = \frac{3da^2}{8\omega}$ is indeed smaller than the maximal value that σ may reach in order that the previous expansion converges as indicated in proposition 2.3.*

Remark 2.2. *We note also that when the stationary solution reaches its maximum amplitude we have $F = \lambda\omega a$ and so we can recover the damping ratio λ from such a forced vibration experiment; this is a close link with the linear case (see for example [GR93] or the English translation [GR97]). This is quite interesting in practice as the damping ratio is usually difficult to measure; we have here a kind of stability result for this experiment.*

2.2.5 Computation of stationary solution

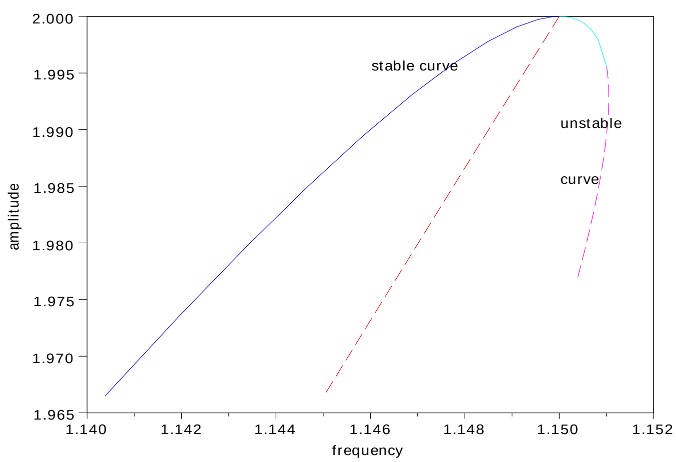

Figure 1: amplitude versus frequency of stationary forced solution in blue and magenta; amplitude of free solution in red

We have numerically solved equation (69) for a range of sigma around the value $\sigma = \frac{3da^2}{8\omega}$ for which the amplitude is maximum; we have chosen $\epsilon = .1; \lambda = 1/2; F = 1; \omega = 1; d = 1$; in figure 1, the solid line shows the solution of this equation that we have solved with several values of sigma using the routine `fsolve` of `Scilab` which implements a modification of the Powell hybrid method. We have noticed in proposition 2.2 that the solution is stable when σ is not too large; indeed the routine `fsolve` fails to solve the equation when we increase too much σ; to go further this point, with the same routine, we have computed various values of sigma for decreasing values of the amplitude; we have plotted this solution with a magenta dotted line. We have added a red dotted line which is the amplitude of the free undamped solution and we notice that it crosses the stationary solution at the point where it reaches its maximum value as stated in previous proposition 2.2.

2.2.6 Dynamic solution

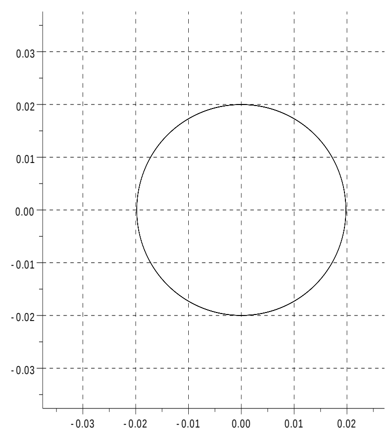

Figure 2: Phase portrait for $u_0 = 0.019796915, \omega_\epsilon = 1.0143379$

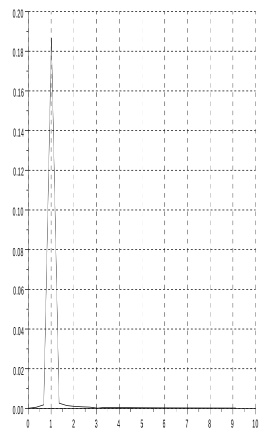

Figure 3: Absolute value of the Fourier transform for $u_0 = 0.019796915, \omega_\epsilon = 1.0143379$

For various values of the initial condition, we compute numerically the solution of (24) with a standard theta method. We use $\epsilon = .01, \lambda = 1/2, F = 1; \omega = 1, d = 1$

In figure 2, we find the phase portrait of the solution with initial values $u(0) = 0.019796915, u'(0) = 0$ so that the angular frequency of the applied force is $\tilde{\omega}_\epsilon = 1.0143379$, we notice that the solution looks periodic (up to the numerical approximation of the method); the initial value of the displacement is computed from a value of a, σ of the stationary solution (69) which is computed in the previous paragraph. The Fourier transform in figure 3 shows only one peak at the angular frequency 1.0143379 which is the angular frequency of the applied force.

In figure 4 , for the same value of the frequency of the applied force, we find the phase portrait of the solution with initial values $u(0) = 0.079, u'(0) = 0$;

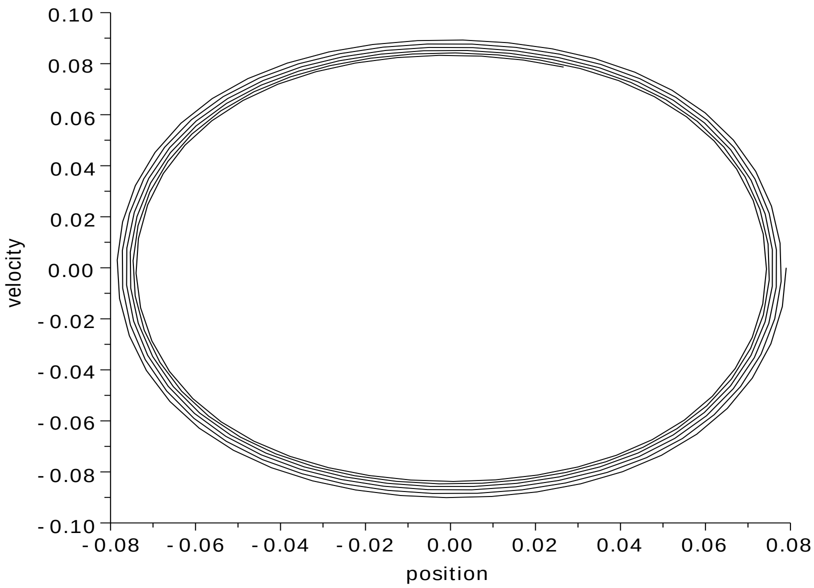

Figure 4: Phase portrait for $u_0 = 0.079, \omega_\epsilon = 1.0143379$

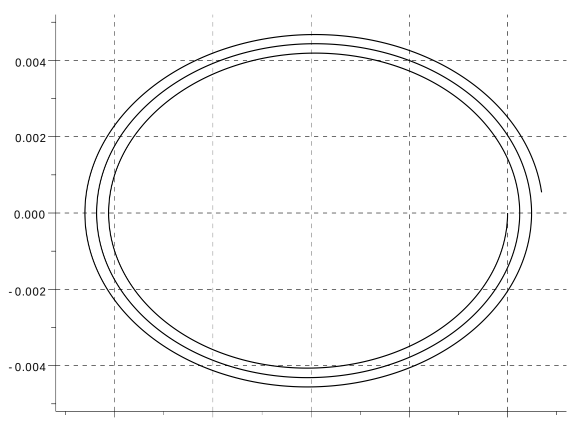

Figure 5: Phase portrait for $u_0 = 0.004, \omega_\epsilon = 1.0143379$

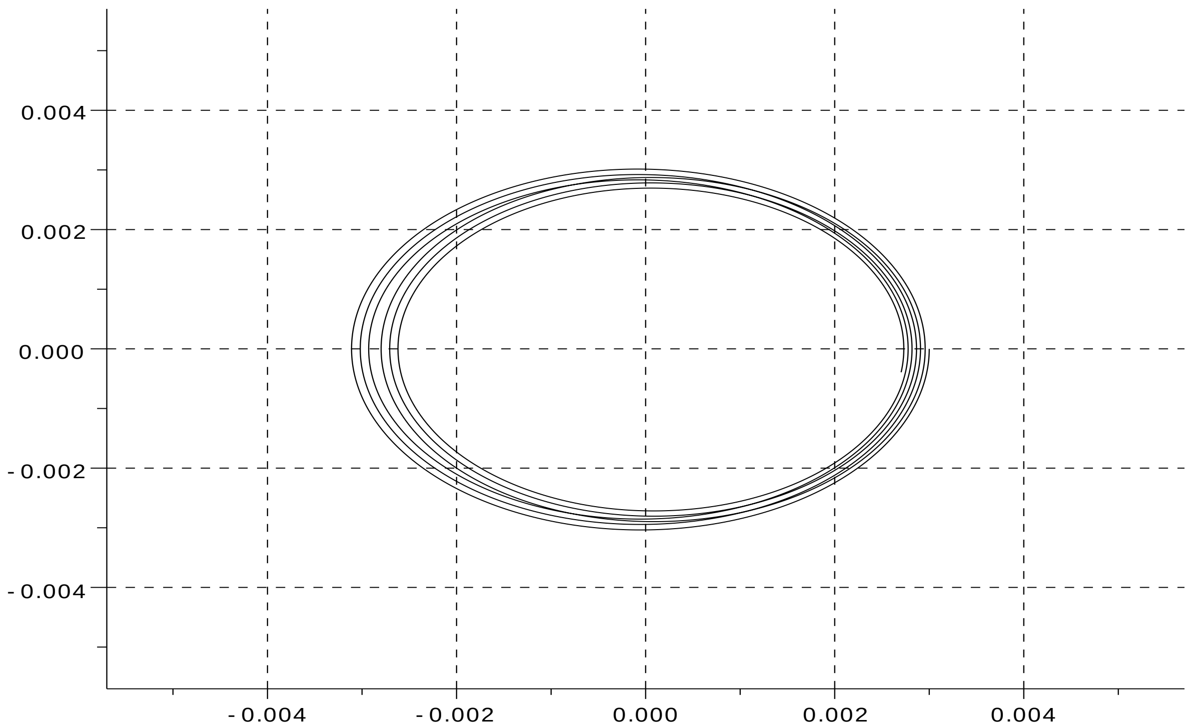

Figure 6: Phase portrait for $u_0 = 0.003, \omega_\epsilon = 0.5$

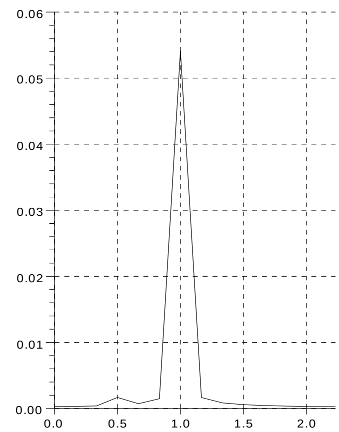

Figure 7: Fourier transform for $u_0 = 0.003, \omega_\epsilon = 0.5$

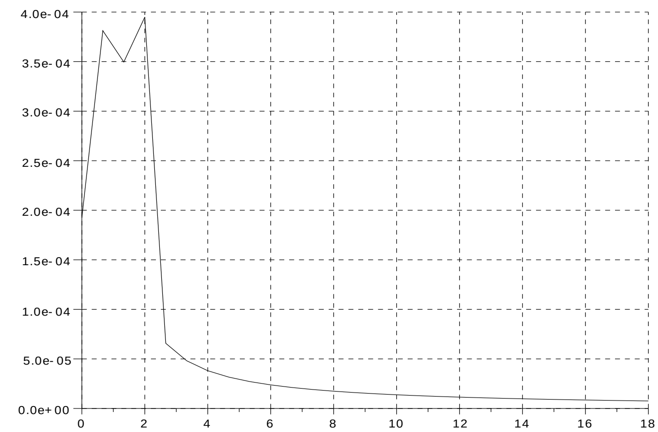

Figure 8: Fourier transform for $u_0 = 0.0001, \omega_\epsilon = 2$

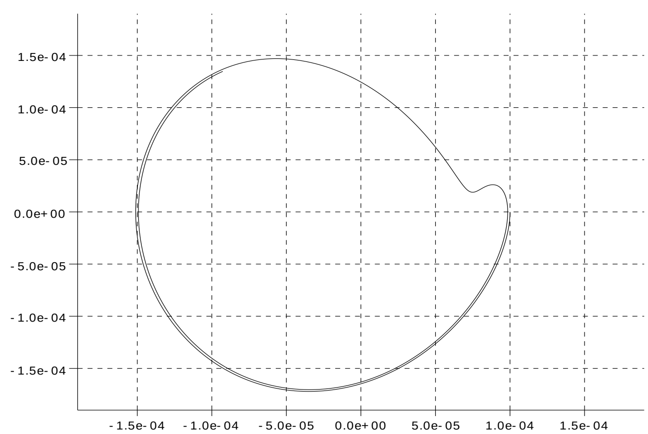

Figure 9: Phase portrait for $u_0 = 0.0001, \omega_\epsilon = 2$

the initial value is larger than the one of the stationary solution and we notice that the solution is decreasing as expected from the stability of the stationary solution.

We find an analogous behavior with an initial value smaller than the stationary solution; in figure 5, for the same value of the frequency of the applied force, we find the phase portrait of the solution with initial values $u(0) = 0.004, u'(0) = 0$; here the solution is increasing as expected from the stability of the stationary solution.

In the case where a, ω_ϵ are far from the stationary curve, we suspect that the frequency content of the response will involve the frequency of the applied force and some frequency due to the system; in figure 6, we find for $a = 0.3, \omega_\epsilon = 0.5$ the phase portrait of the solution, the frequency transform is in figure 7; on this plot of the Fourier transform, we notice two peaks icluding the angular frequency $\omega_\epsilon = 0.5$ of the applied load.

For large values of ω_ϵ, the phase portrait is less regular see figure 9 for $a = 0.01, \omega_\epsilon = 2$; we find also two peaks for the Fourier transform in figure 8.

3 System with a strong local cubic non linearity

In the previous section, we have derived a double scale expansion of a solution of a one degree of freedom free vibrations system and damped vibrations with sinusoidal forcing with frequency close to free vibration frequency. Now, we extend the results to the case of multiple degrees of freedom.

3.1 Free vibrations, double scale expansion

We consider a system of vibrating masses attached to springs:

$$M\ddot{u} + Ku + \Phi(u,\epsilon) = 0. \tag{71}$$

The mass matrix M and the rigidity matrix K are assumed to be symmetric and positive definite. See an example in section 3.2.5 We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the endpoints of some spring assumed to be non linear:

$$\Phi_{p-1}(u,\epsilon) = c(u_p - u_{p-1})^2 + \frac{d}{\epsilon}(u_p - u_{p-1})^3,\ \Phi_p = -\Phi_{p-1},\quad p = 2,\ldots,n \tag{72}$$

If the non linear spring would have been the first or the last one, the expression of the function Φ would depend on the boundary condition; each case would be solved using the same method with slight changes in some formulas. In order to get an approximate solution, we are going to write it in the generalized eigenvector basis:

$$K\phi_k = \omega_k M\phi_k,\ \text{with}\ \phi_k^T M\phi_l = \delta_{kl},\quad k,l = 1\ldots,n. \tag{73}$$

So we perform the change of function

$$u = \sum_{k=1}^{n} y_k\phi_k \tag{74}$$

we obtain

$$\ddot{y}_k + \omega_k^2 y_k + \phi_k^T\Phi(\sum_{i=1}^{n} y_i\phi_i,\epsilon) = 0,\quad k = 1\ldots,n. \tag{75}$$

As Φ has only 2 components which are not zero, it can be written

$$\ddot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p})\,\Phi_{p-1}(\sum_{i=1}^{n} y_i\phi_i,\epsilon) = 0,\quad k = 1\ldots,n \tag{76}$$

or more precisely

$$\ddot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p})\left[c\left(\sum_{i=1}^{n} y_i(\phi_{i,p} - \phi_{i,p-1})\right)^2 + \right.$$
$$\left.\frac{d}{\epsilon}\left(\sum_{i=1}^{n} y_i(\phi_{i,p} - -\phi_{i,p-1})\right)^3\right] = 0,\quad k = 1\ldots,n. \tag{77}$$

As for the 1 d.o.f. case, we use a double scale expansion to compute an approximate small solution; more precisely, we look for a solution close to the normal mode of the associated linear system; we denote this mode by subscript 1; obviously by permuting the coordinates, this subscript could be anyone (different of p, this case would give similar results with slightly different formulas); we set

$$T_0 = \omega_1 t, \quad T_1 = \epsilon t \tag{78}$$

and we use the *ansatz*:

$$y_k = \epsilon y_k^1(T_0, T_1) + \epsilon^2 r_k(T_0, T_1, \epsilon) \tag{79}$$

so that

$$\frac{d^2 y_k}{dt^2} = \epsilon\omega_1^2 D_0^2 y_k^1 + \epsilon^2[2\omega_1 D_0 D_1 y_k^1 + \omega_1^2 D_0^2 r_k] + \epsilon^3[D_1^2 y_k^1 + \mathcal{D}_2 r_k] \tag{80}$$

with

$$\mathcal{D}_2 r_k = \frac{1}{\epsilon}\left(\frac{d^2 r_k}{dt^2} - \omega_1^2 D_0^2 r_k\right) = 2\omega_1 D_0 D_1 r_k + \epsilon D_1^2 r_k. \tag{81}$$

We plug previous expansions into (77). By identifying the coefficients of the powers of ϵ in the expansion of (76), we get:

$$\left\{\begin{array}{l} \omega_1^2 D_0^2 y_k^1 + \omega_k^2 y_k^1 = 0,\ k = 1\ldots, n \\ \omega_1^2 D_0^2 r_k + \omega_k^2 r_k = S_{2,k},\ k = 1\ldots, n \quad \text{with} \end{array}\right. \tag{82}$$

to simplify, the manipulations, we set

$$\delta_p\phi_l = (\phi_{l,p} - \phi_{l,p-1}),$$

so:

$$S_{2,k} = \frac{-\delta_p\phi_k}{\epsilon^2}\Phi_{p-1}\left(\sum_i(\epsilon y_i^1 + \epsilon^2 r_i)\phi_i, \epsilon\right) - 2\omega_1 D_0 D_1 y_k^1 - \epsilon\mathcal{R}_k \tag{83}$$

with

$$\mathcal{R}_k = \left(D_1^2 y_k^1 + \mathcal{D}_2 r_k\right) \tag{84}$$

and

$$S_{2,k} = \frac{-\delta_p\phi_k}{\epsilon^2}\left[c\left(\sum_i(\epsilon y_i^1 + \epsilon^2 r_i)\delta_p\phi_i\right)^2 + \frac{d}{\epsilon}\left(\sum_i(\epsilon y_i^1 + \epsilon^2 r_i)\delta_p\phi_i\right)^3\right] - 2\omega_1 D_0 D_1 y_k^1 - \epsilon\mathcal{R}_k. \tag{85}$$

The formula may be expanded

$$S_{2,k} = -\delta_p\phi_k\left[c\sum_{i,j} y_i^1 y_j^1 \delta_p\phi_i\delta_p\phi_j + d\sum_{i,j,l} y_i^1 y_j^1 y_l^1 \delta_p\phi_i\delta_p\phi_j\delta_p\phi_l\right] - 2\omega_1 D_0 D_1 y_k^1 - \epsilon R_k\left(y^1, r, \epsilon\right) \tag{86}$$

where

$$R_k(y^1, r, \epsilon) = \mathcal{R}_k + \delta_p\phi_k\left[\epsilon c \sum_{i,j}\left(2y_i^1 r_j + \epsilon r_i r_j\right)\delta_p\phi_i\delta_p\phi_j + \epsilon d \sum_{ijl}\left(3y_i^1 y_j^1 r_l + 3\epsilon y_i^1 r_j r_l + 3\epsilon^2 r_i r_j r_l\right)\delta_p\phi_i\delta_p\phi_j\delta_p\phi_l\right]. \quad (87)$$

We set $\theta(T_0, T_1) = T_0 + \beta(T_1)$ and we note that $D_0\theta = 1,\ D_1\theta = D_1\beta$; we solve the first set of equations (82), imposing $O(\epsilon)$ initial Cauchy data for $k \neq 1$; we get:

$$y_1^1 = a(T_1)cos(\theta), \quad \text{and } y_k^1 = O(\epsilon),\ k = 2\dots n \quad (88)$$

we put terms involving $y_k^1,\ k \geq 2$ into R_k; so we obtain

$$S_{2,1} = -\delta_p\phi_1\left[c\left(y_1^1\delta_p\phi_1\right)^2 + d\left(y_1^1\delta_p\phi_1\right)^3\right] - 2\omega_1 D_0 D_1 y_1^1 - \epsilon R_1(y^1, r, \epsilon) \text{ and} \quad (89)$$

$$S_{2,k} = -\delta_p\phi_k\left[c\left(y_1^1\delta_p\phi_1\right)^2 + d\left(y_1^1\delta_p\phi_1\right)^3\right] - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \quad (90)$$

Using (88), we get:

$$S_{2,1} = -\delta_p\phi_1\Big[\frac{ca_1^2}{2}(1 + cos(2\theta))\left(\delta_p\phi_1\right)^2 + \frac{da_1^3}{4}\left((cos(3\theta) + 3cos(\theta))(\delta_p\phi_1)^3\right)\Big] + 2\omega_1(D_1 a_1 sin(\theta) + a_1 D_1\beta_1 cos(\theta)) - \epsilon R_1(y^1, r, \epsilon) \text{ and} \quad (91)$$

$$S_{2,k} = -\delta_p\phi_k\Big[\frac{ca_1^2}{2}(1 + cos(2\theta))\left(\delta_p\phi_1\right)^2 + \frac{da_1^3}{4}\left((cos(3\theta) + 3cos(\theta))(\delta_p\phi_1)^3\right)\Big] + - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \quad (92)$$

We gather the terms at angular frequency 1 in $S_{2,1}$

$$S_{2,1} = -\delta_p\phi_1\Big[\frac{da_1^3}{4}3cos(\theta)(\delta_p\phi_1)^3\Big] + 2\omega_1(D_1 a_1 sin(\theta) + a_1 D_1\beta_1 cos(\theta)) + S_{2,1}^\sharp - \epsilon R(y^1, r, \epsilon) \quad (93)$$

with

$$S^{\sharp}_{2,1} \quad = \quad -\delta_p\phi_1\left[\frac{ca_1^2}{2}(1+cos(2\theta))(\delta_p\phi_1)^2+\frac{da_1^3}{4}cos(3\theta)(\delta_p\phi_1)^3\right]. \quad (94)$$

If we enforce

$$D_1 a_1 = 0, \quad \text{and} \quad 2\omega_1 a_1 D_1\beta_1 = (\delta_p\phi_1)^4\frac{3da^3}{4} \text{ so that}$$
$$a_1 = a_{1,0}, \quad \beta_1 = \beta_{1,0}T_1 \text{ with } \beta_{1,0} = \frac{3da^2}{8\omega}(\delta_p\phi_1)^4 T_1 \quad (95)$$

the right hand side

$$S_{2,1} = S^{\sharp}_{2,1} - \epsilon R_1(y^1, r, \epsilon) \quad (96)$$

contains no term at angular frequency 1; for the other components, without any manipulation, there is no trouble with the frequencies if we assume that all the eigenfrequencies ω_k for $k = 2\ldots n$ are not multiple of ω_1 ($\omega_k \neq q\omega_1$ for $q = 1$ or $q = 2$, $q = 3$).

In order to prove that r is bounded, after the elimination of terms at frequency 1, we write back the equations with the variable t, for the second set of equations of (82).

$$\omega_1^2\ddot{r}_k + \omega_k^2 r_k = \tilde{S}_{2,k} \qquad \text{for } k = 1,\ldots n \quad (97)$$

with

$$\tilde{S}_{2,1} = S^{\sharp}_{2,1} - \epsilon\tilde{R}_1(y^1, r, \epsilon) \quad (98)$$

where

$$S^{\sharp}_{2,1} = -\delta_p\phi_1\left[\frac{ca_1^2}{2}(1+cos(2(\omega_1 t+\beta_{1,0}\epsilon t))(\delta_p\phi_1)^2 + \frac{da_1^3}{4}cos(3(\omega_1 t+\beta_{1,0}\epsilon t))(\delta_p\phi_1)^3\right] \quad (99)$$

and

$$\tilde{S}_{2,k} = -\delta_p\phi_k\left[\frac{ca_1^2}{2}(1+cos(2(\omega_1 t+\beta_{1,0}\epsilon t)))\,(\delta_p\phi_1)^2 + \frac{da_1^3}{4}\left((cos(3(\omega_1 t+\beta_{1,0}\epsilon t)) + 3cos((\omega_1 t+\beta_{1,0}\epsilon t)))(\delta_p\phi_1)^3\right)\right] - \epsilon\tilde{R}_k(y^1, r, \epsilon) \text{ for } k \neq 1 \quad (100)$$

and where

$$\tilde{R}_k(y^1, r, \epsilon) = R_k(y^1, r, \epsilon) - \mathcal{D}_2 r_k \quad (101)$$

Proposition 3.1. *Under the assumption that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$, there exists $\gamma > 0$ such that for all $t \leq t_\epsilon = \frac{\gamma}{\epsilon}$, the solution of* (76) *with initial data*

$$y_1(0) = \epsilon a_{1,0}, \quad \dot{y}_1(0) = 0, \quad y_k(0) = O(\epsilon^2), \quad \dot{y}_k(0) = 0 \tag{102}$$

satisfy the following expansion

$$y_1(t) = \epsilon a_0 cos(\nu_\epsilon t) + \epsilon^2 r_1(\epsilon, t) \text{ with } \nu_\epsilon = \omega_1 + 3\epsilon \frac{da_0^2}{8\omega_1}(\phi_{1,p} - \phi_{1,p-1})^4 \tag{103}$$

$$y_k(t) = \epsilon^2 r_k(\epsilon, t) \tag{104}$$

with r_k uniformly bounded in $\mathcal{C}^2(0, t_\epsilon)$ for $k = 1, \dots n$ and ω_1, ϕ_1 are the eigenvalue and eigenvectors defined in (73)*.*

Corollary 3.1. *The solution of* (71)*,* (72) *with*

$$\phi_1^T u(0) = \epsilon a_{1,0}, \quad \phi_1^T \dot{u}(0) = 0, \quad \phi_k^T u(0) = O(\epsilon^2), \quad \phi_k^T \dot{u}(0) = 0$$

with ω_k, ϕ_k are the eigenvalue and eigenvectors defined in (73)

$$is \quad u(t) = \sum_{k=1}^{n} y_k(t)\phi_k \tag{105}$$

with the expansion of y_k of previous proposition.

Proof. For the proposition, we use lemma 5.4. Set $S_1 = S_{2,1}^\sharp, \quad S_k = S_{2,k}$ for $k = 1, \dots n$; as we have enforced (95), the functions S_k are periodic, bounded, and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$; so $S_k, \ k = 1, \dots, n$ satisfies the lemma hypothesis. Similarly, set $g = \tilde{R}$, its components are polynomials in r with coefficients which are bounded functions, so it is lipschitzian on the bounded subsets of $\mathbb{R}$, it satisfies the hypothesis of lemma 5.4 and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (107) □

Remark 3.1. *We have obtained a periodic asymptotic expansion of a solution of system* (71)*,* (72)*; they are called non linear normal modes in the mechanical community ([KPGV09, JPS04]. In the next section, we shall derive that the frequencies of the normal mode are resonant frequencies for an associated forced system, the so called primary resonance; secondary resonance could be derived along similar lines.*

3.2 Forced, damped vibrations, double scale expansion

3.2.1 Derivation of the expansion

We consider a similar system of forced vibrating masses attached to springs with a light damping:

$$M\ddot{u} + \epsilon C\dot{u} + Ku + \Phi(u, \epsilon) = \epsilon^2 F cos(\tilde{\omega}_\epsilon t) \tag{106}$$

with the same assumptions as in subsection 3.1. We assume that the frequency of the driving force is close to some frequency of the linearised system (primary resonance); we denote this frequency with the subscript 1: $\tilde{\omega}_\epsilon = \omega_1 + \epsilon\sigma$.

We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the endpoints of some spring assumed to be non linear. As for free vibrations, we perform the change of function

$$u = \sum_{k=1}^{n} y_k \phi_k \tag{107}$$

with ϕ_k, the generalised eigenvectors of (73). As the damping matrix C is usually not well defined, to simplify, we assume that it is diagonal in the eigenvector basis $\phi_k,\ k = 1, \dots n$. We obtain

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + \phi_k^T \Phi(\sum_{i=1}^{n} y_i \phi_i, \epsilon) = \epsilon^2 f_k cos(\tilde{\omega}_\epsilon t), \quad k = 1 \dots, n \tag{108}$$

with $f_k = \phi_k^T F$. As for the free vibration case, Φ has only 2 components which are not zero, so the system can be written

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}, \epsilon)\, \Phi_{p-1}(\sum_{i=1}^{n} y_i \phi_i) = \epsilon^2 f_k cos(\tilde{\omega}_\epsilon t), \quad k = 1 \dots, n \tag{109}$$

or more precisely

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \frac{d}{\epsilon} \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^3 \right] = \epsilon^2 f_k cos(\tilde{\omega}_\epsilon t), \quad k = 1 \dots, n. \tag{110}$$

As for the 1 d.o.f. case, we use a double scale expansion to compute an approximate small solution; we use a fast scale which is ϵ dependent; we set

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t \tag{111}$$

and we use the *"ansatz"*

$$y_k = \epsilon y_k^1(T_0, T_1) + \epsilon^2 r_k(T_0, T_1, \epsilon) \tag{112}$$

so that

$$\frac{dy_k}{dt} = \epsilon \big[\omega_1 D_0 y_k^1 + \epsilon\sigma D_0 y_k^1 + \epsilon D_1 y_k^1 \big] + \epsilon^2 \omega_1 D_0 r_k + \epsilon^2 (\frac{dr_k}{dt} - \omega_1 D_0 r_k) \tag{113}$$

$$\frac{d^2 y_k}{dt^2} = \epsilon \Big\{ \omega_1^2 D_0^2 y_k^1 + 2\epsilon\omega_1 \left[\sigma D_0^2 y_k^1 + D_0 D_1 y_k^1\right] + \epsilon^2 \left[\sigma^2 D_0^2 y_k^1 + 2\sigma D_0 D_1 y_k^1 + D_1^2 y_k^1\right] \Big\} + \epsilon^2 \omega_1^2 D_0^2 r_k + \epsilon^3 \mathcal{D}_2 r_k \quad (114)$$

with

$$\mathcal{D}_2 r_k = \frac{1}{\epsilon}\left(\frac{d^2 r_k}{dt^2} - \omega_1^2 D_0^2 r_k\right) = 2\omega_1(\sigma D_0^2 r_k + D_0 D_1 r_k) + \epsilon\left[\sigma^2 D_0^2 r_k + 2\sigma D_0 D_1 r_k + D_1^2 r_k\right]. \quad (115)$$

We plug previous expansions into (110). By identifying the coefficients of the powers of ϵ in the expansion of (110), we get:

$$\left\{ \begin{array}{l} \omega_1^2 D_0^2 y_k^1 + \omega_k^2 y_k^1 = 0,\ k = 1\ldots, n \\ \omega_1^2 D_0^2 r_k + \omega_k^2 r_k = S_{2,k},\ k = 1\ldots, n \quad \text{with} \end{array} \right. \quad (116)$$

$$S_{2,k} = -\left\{ \frac{\delta_p \phi_k}{\epsilon^2} \Phi_{p-1}\left(\sum_i (\epsilon y_i^1 + \epsilon^2 r_i)\phi_i, \epsilon\right) + 2\omega_1[D_0 D_1 y_k^1 + \sigma D_0^2 y_k^1] + \lambda_k \omega_1 D_0 y_k^1 \right\} + f_k cos(T_0) - \epsilon R_k(y^1, r, \epsilon) \quad (117)$$

where we gather higher order terms in R_k and to simplify, the manipulations, we have set

$$\delta_p \phi_l = (\phi_{l,p} - \phi_{l,p-1}),$$

so:

$$S_{2,k} = -\frac{\delta_p \phi_k}{\epsilon^2}\left[c\left(\sum_i (\epsilon y_i^1 + \epsilon^2 r_i)\delta_p \phi_i\right)^2 + \frac{d}{\epsilon}\left(\sum_i (\epsilon y_i^1 + \epsilon^2 r_i)\delta_p \phi_i\right)^3 \right] - 2\omega_1[D_0 D_1 y_k^1 + \sigma D_0^2 y_k^1] - \lambda_k \omega_1 D_0 y_k^1 + f_k cos(T_0) - \epsilon R_k(y^1, r, \epsilon). \quad (118)$$

The formula may be expanded

$$S_{2,k} = -\delta_p \phi_k \left[c \sum_{i,j} y_i^1 y_j^1 \delta_p \phi_i \delta_p \phi_j + d \sum_{i,j,l} y_i^1 y_j^1 y_l^1 \delta_p \phi_i \delta_p \phi_j \delta_p \phi_l \right] - 2\omega_1[D_0 D_1 y_k^1 + \sigma D_0^2 y_k^1] - \lambda_k \omega_1 D_0 y_k^1 + f_k cos(T_0) - \epsilon R_k(y^1, r, \epsilon) \quad (119)$$

We set $\theta(T_0, T_1) = T_0 + \beta(T_1)$ and we note that $D_0\theta = 1,\ D_1\theta = D_1\beta$; we solve the first set of equations (116), imposing initial Cauchy data for $k \neq 1$ of order $O(\epsilon)$ we get:

$$y_1^1 = a_1(T_1)cos(\theta), \quad \text{and } y_k^1 = O(\epsilon),\ k = 2 \dots n \tag{120}$$

we put terms involving y_k^1 into R_k for $k \geq 2$ and so we obtain

$$S_{2,1} = -\delta_p\phi_1 \left[c\left(y_1^1\delta_p\phi_1\right)^2 + d\left(y_1^1\delta_p\phi_1\right)^3\right] \\ - 2\omega_1[D_0D_1y_1^1 + \sigma D_0^2 y_1^1] - \lambda_1\omega_1 D_0 y_1^1 + f_1\cos(T_0) - \epsilon R_1(y^1, r, \epsilon) \text{ and} \tag{121}$$

$$S_{2,k} = -\delta_p\phi_k \left[c\left(y_1^1\delta_p\phi_1\right)^2 + d\left(y_1^1\delta_p\phi_1\right)^3\right] + \\ f_k\cos(T_0) - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \tag{122}$$

Using (120), we get:

$$S_{2,1} = -\delta_p\phi_1 \Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\left(\delta_p\phi_1\right)^2 + \\ \frac{da_1^3}{4}\left((\cos(3\theta) + 3\cos(\theta))(\delta_p\phi_1)^3\right)\Big] + \\ 2\omega_1[D_1a_1 sin(\theta) + a_1D_1\beta_1\cos(\theta) + \sigma a_1\cos(\theta)] + \lambda_1\omega_1 a_1 sin(\theta) \\ + f_1[\sin(\theta)\sin(\beta) + \cos(\theta)\cos(\beta)] - \epsilon R_1(y^1, r, \epsilon) \text{ and} \tag{123}$$

$$S_{2,k} = -\delta_p\phi_k \Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\left(\delta_p\phi_1\right)^2 + \\ \frac{da_1^3}{4}\left((\cos(3\theta) + 3\cos(\theta))(\delta_p\phi_1)^3\right)\Big] + \\ + f_k[\sin(\theta)\sin(\beta) + \cos(\theta)\cos(\beta)] - \epsilon R_k(y^1, r, \epsilon) \text{ for } k \neq 1. \tag{124}$$

We gather the terms at angular frequency 1 in $S_{2,1}$

$$S_{2,1} = \delta_p\phi_1 \left[-3\frac{da_1^3}{4}\cos(\theta)(\delta_p\phi_1)^3 + 2\omega_1(a_1D_1\beta_1 + \sigma a_1) + f_1\cos(\beta)\right]\cos(\theta) \\ + \left[\omega_1(2D_1a_1 + \lambda_1 a_1) + f_1 sin(\beta)\right]sin(\theta) + S_{2,1}^\sharp - \epsilon R(y^1, r, \epsilon) \tag{125}$$

with

$$S_{2,1}^\sharp \quad = \quad -\delta_p\phi_1\left[\frac{ca_1^2}{2}(1 + \cos(2\theta))(\delta_p\phi_1)^2 + \frac{da_1^3}{4}\cos(3\theta)(\delta_p\phi_1)^3\right]. \tag{126}$$

Orientation If we enforce

$$\left\{\begin{array}{rl} \omega_1\big(2D_1a_1+\lambda_1a_1\big) & = -f_1sin(\beta_1), \quad \text{and} \\ 2\omega_1\big(a_1D_1\beta_1+\sigma a_1\big) & = \frac{3da^3}{4}(\delta_p\phi_1)^4 - f_1cos(\beta_1) \end{array}\right. \tag{127}$$

the right hand side

$$S_{2,1} = S^\sharp_{2,1} - \epsilon R_1(y^1, r, \epsilon) \tag{128}$$

contains no term at angular frequency 1; for the other components, without any manipulation, there is not such terms , if we assume that all the eigenfrequencies ω_k for $k = 2\dots n$ are not multiple of ω_1 ($\omega_k \neq q\omega_1$ for $q = 1$ or $q = 2$, $q = 3$). This will enable us to justify this expansion; previously, we study the stationary solution of this approximate system and the stability of the solution in a neighbourhood of this stationary solution.

3.2.2 Stationary solution and stability

The situation is very close to the 1 d.o.f. case; except the replacement of d by of $\tilde{d} = d(\delta_p\phi_1)^4$, the system (127) is the same as (51); the other components are zero. We state a similar proposition

Proposition 3.2. *When $\sigma \leq \frac{3\tilde{d}\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9\tilde{d}^2\bar{a}^4}{16\omega^2} - \lambda_1^2}$, the stationary solution of* (127) *is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary one, it remains close and converges to it); to the stationary case corresponds the approximate solution of* (77) $y^1_1 = \bar{a}_1\cos(T_0 + \bar{\beta}_1)$, $y^1_k = O(\epsilon)$, $k = 2, \dots, n$, *it is periodic; for an initial data close enough to the stationary solution,* $y^1_1 = a(T_1)\cos(T_0 + \beta_1(T_1))$, $y^1_k = O(\epsilon)$, $k = 2, \dots, n$ *with* a, β_1 *solutions of* (127) *with d replaced by $\tilde{d}$; they converge to the stationary solution $\bar{a}_1, \bar{\beta}_1$ when $T_1 \longrightarrow +\infty$.*

3.2.3 Convergence of the expansion

In order to prove that r is bounded, after the elimination of terms at frequency 1, we write back the equations with the variable t, for the second set of equations of (82).

$$\omega_1^2\ddot{r}_k + \omega_k^2 r_k = \tilde{S}_{2,k} \quad \text{for } k = 1, \dots n \tag{129}$$

with

$$\tilde{S}_{2,1} = S^\sharp_{2,1} - \epsilon\tilde{R}_1(y^1, r, \epsilon) \tag{130}$$

where

$$S^\sharp_{2,1} = -\delta_p\phi_1\Bigg[\frac{c(a_1(\epsilon t))^2}{2}(1 + \cos(2(\tilde{\omega}_\epsilon t + \beta_1(\epsilon t))(\delta_p\phi_1)^2 \\ + \frac{da_1^3}{4}\cos(3(\tilde{\omega}_\epsilon t + \beta_1(\epsilon t))(\delta_p\phi_1)^3\Bigg] \tag{131}$$

and

$$S_{2,k} = -\delta_p\phi_k\left[\frac{c(a_1(\epsilon t)^2}{2}(1+\cos(2(\tilde{\omega}_\epsilon t+\beta_1(\epsilon t)))\left(\delta_p\phi_1\right)^2 + \right.$$

$$\left.\frac{da_1^3}{4}\left((\cos(3(\tilde{\omega}_\epsilon t+\beta_(\epsilon t))+3\cos((\tilde{\omega}_\epsilon t+\beta_1(\epsilon t)))(\delta_p\phi_1)^3\right)\right]$$

$$-\epsilon R_k(y^1,r,\epsilon) \text{ for } k\neq 1 \tag{132}$$

where

$$\tilde{R}_k(y^1,r,\epsilon) = R_k(y^1,r,\epsilon) - \mathcal{D}_2 r_k - \lambda_k\big(\frac{dr_k}{dt} - \omega_k D_0 r_k\big) \tag{133}$$

Proposition 3.3. *Under the assumption that ω_k and ω_1 are $\mathbb{Z}$ independent for $k\neq 1$, there exists $\gamma>0$ such that for all $t\le t_\epsilon = \frac{\gamma}{\epsilon}$, the solution of* (110) *with initial data*

$$y_1(0) = \epsilon a_{1,0} + O(\epsilon^2), \quad \dot{y}_1(0) = -\epsilon\omega a_{1,0} sin(\beta_{1,0}) + O(\epsilon^2), \tag{134}$$

$$y_k(0) = O(\epsilon^2), \quad \dot{y}_k(0) = 0 \tag{135}$$

and with the initial data close to the stationary solution

$$|a_{1,0} - \bar{a}_1| \le \epsilon C_1, \quad |\beta_{1,0} - \bar{\beta}_1| \le \epsilon C_1$$

satisfy the following expansion

$$y_1(t) = \epsilon a_1(\epsilon t)\cos(\tilde{\omega}_\epsilon t + \beta_1(\epsilon t)) + \epsilon^2 r_1(\epsilon,t) \text{ with} \tag{136}$$

$$y_k(t) = \epsilon^2 r_k(\epsilon,t) \tag{137}$$

with a_1,β_1 solution of (127) *and with r_k uniformly bounded in $\mathcal{C}^2(0,t_\epsilon)$ for $k=1,\dots n$ and ω_1,ϕ_1 are the eigenvalue and eigenvectors defined in* (73) *and $a_1\beta_1$ are solution of* (127)

Corollary 3.2. *The solution of* (106), (72) *with*

$$\phi_1^T u(0) = \epsilon a_{1,0}, \quad \phi_1^T \dot{u}(0) = -\epsilon\omega_1 a_{1,0}\sin(\beta_{1,0}), \quad \phi_k^T u(0) = O(\epsilon^2), \quad \phi_k^T\dot{u}(0) = 0$$

with ω_k,ϕ_k the eigenvalues and eigenvectors defined in (73).

$$\textit{is} \quad u(t) = \sum_{k=1}^n y_k(t)\phi_k \tag{138}$$

with the expansion of y_k of previous proposition.

Proof. For the proposition, we use lemma 5.4. Set $S_1 = S_{2,1}^\sharp,\ \ S_k = S_{2,k}$ for $k=1,\dots n$; as we have enforced (95), the functions S_k are periodic, bounded, and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k\neq 1$; so S satisfies the lemma hypothesis. Similarly, set $g=\tilde{R}$, it is a polynomial in r with coefficients which are bounded functions , so it is lipschitzian on the bounded subsets of $\mathbb{R}$, it satisfies the hypothesis of lemma 5.4 and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (107) □

3.2.4 Maximum of the stationary solution

As equation (127) is similar to the equation (51) of the 1 d.o.f. case, we get also that the stationary solution reaches its maximum amplitude to the frequency of the free periodic solution.

Consider the stationary solution of (127), it satisfies

$$\left\{\begin{array}{rl} \lambda_1 a_1 \omega_1 & = -f_1 \sin(\beta_1) \\ a\left(2\omega_1\sigma - \frac{3\tilde{d}a^2}{4}\right) & = -f_1 \cos(\beta_1) \end{array}\right. \tag{139}$$

manipulating, we get that a_1 is solution of the equation:

$$f(a_1,\sigma) = \lambda_1^2 a_1^2 \omega^2 + a_1^2 \left(2\omega_1\sigma - \frac{3\tilde{d}a_1^2}{4}\right)^2 - f_1^2 = 0. \tag{140}$$

As for the 1 d.o.f. case, we can state:

Proposition 3.4. *The stationary solution of* (127) *satisfies*

$$\left\{\begin{array}{rl} \lambda_1 a_1 \omega_1 & +f_1 \sin(\beta_1) = 0 \\ 2a_1\omega_1\sigma - \frac{3\tilde{d}a^3}{4} & +f_1 \cos(\beta_1) = 0 \end{array}\right. \tag{141}$$

it reaches its maximum amplitude for $\sigma = \frac{3\tilde{d}a_1^2}{8\omega_1}$ and $\beta_1 = \frac{\pi}{2} + k\pi$; the excitation is at the frequency

$$\tilde{\omega}_\epsilon = \omega_1 + 3\epsilon\frac{\tilde{d}a_1^2}{8\omega_1}, \quad \textit{with} \quad \tilde{d} = d(\Phi_{1,p} - \Phi_{1,p-1})^4 \quad \textit{and} \quad F = \lambda_1\omega_1 a_1$$

where Φ_1 is the eigenvector of the underlying linear system associated to ω_1; $\tilde{\omega}_\epsilon$ is the frequency of the free periodic solution (23)*; for this frequency, the approximation (of the solution up to the order ϵ) is periodic:*

$$y_1(t) = \epsilon\frac{f_1}{\lambda_1\omega_1}\sin(\tilde{\omega}_\epsilon t) + \epsilon^2 r(\epsilon,t) \tag{142}$$

$$y_k(t) = \epsilon^2 r_k(\epsilon,t) \tag{143}$$

As for the 1 d.o.f. case we can remark the following points.

Remark 3.2. *This value of $\sigma = \frac{3\tilde{d}a_1^2}{8\omega_1}$ is indeed smaller than the maximal value that σ may reach in order that the system be stable and that the previous expansion converges as indicated in proposition 2.3.*

Remark 3.3. *We note also that when the stationary solution reaches its maximum amplitude we have $f_1 = \lambda_1\omega_1 a_1$ and so we can recover the damping ratio λ_1 from such a forced vibration experiment; this is a close link with the linear case (see for example [GR93] or the English translation [GR97]). This is quite interesting in practice as the damping ratio is usually difficult to measure. Obviously, we can recover the damping ratio for other frequencies by performing other experiments.*

We can also consider this result as a stability of the process used in the linear case with respect to the appearance of a small non-linearity.

3.2.5 Numerical solution

We consider numerical solution of (106) with (72); we have chosen $M = I$; $u = 0$ at both ends, so K is the classical matrix

$$k \begin{pmatrix} 2 & -1 & \ldots\ldots\ldots\ldots & & \\ -1 & 2 & -1 & \ldots\ldots\ldots & \\ 0 & -1 & 2 & -1 & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots\ldots & & & -1 & 2 \end{pmatrix};$$

$C = \lambda I$ with $\lambda = 1/2$; for numerical balance, we have computed $\frac{u}{\epsilon}$; with the choice $p = 1$ we have $\Phi_1 = \epsilon[cu_1^2 + du_1^3]$ with $c = 1, d = 1$. In figure 10, we find 3 curves in phase space for components $1, 3, 6$ of the system. In figure 11, we find the Fourier transform of the components; some components have the same transform; the graphs are slightly non symmetric.

4 Conclusion

For differential systems modeling spring-masses vibrations with non linear springs, we have derived and rigorously proved a double scale analysis of periodic solution of free vibrations (so called non linear normal modes); for damped vibrations with periodic forcing with frequency close to free vibration frequency (the so called primary resonance case), we have obtained an asymptotic expansion and derived that the amplitude is maximal at the frequency of the non linear normal mode. Such non linear vibrating systems linked to a bar generate acoustic waves; an analysis of the dilatation of a one-dimensional nonlinear crack impacted by a periodic elastic wave, a smooth model of the crack may be carried over with a delay differential equation, [jl09].

Acknowledgment We thank S. Junca for his stimulating interest.

5 Appendix

5.1 Inequalities for differential equations

Lemma 5.1. *Let w_ϵ be solution of*

$$w" + w = S(t, \epsilon) + \epsilon g(t, w, \epsilon) \tag{144}$$

$$w(0) = 0, \quad w'(0) = 0. \tag{145}$$

If the right hand side satisfies the following conditions

1. *S is a sum of periodic bounded functions:*
   (a) *for all t and for all ϵ small enough, $S(t, \epsilon) \leq M$*

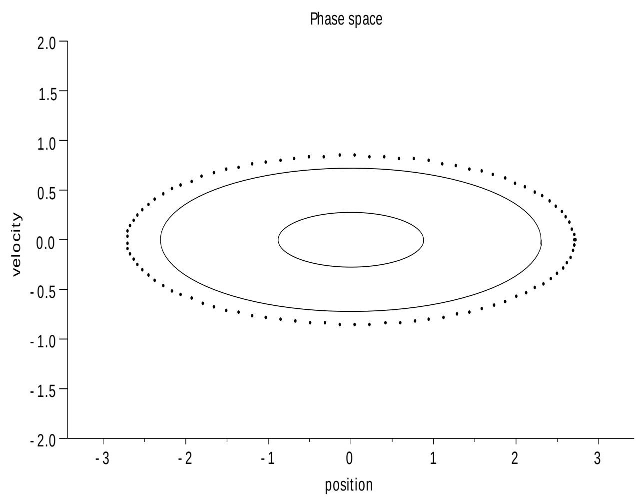

Figure 10: Phase portrait of a system with 9 d.o.f. for $\omega_\epsilon = 0.3128868$

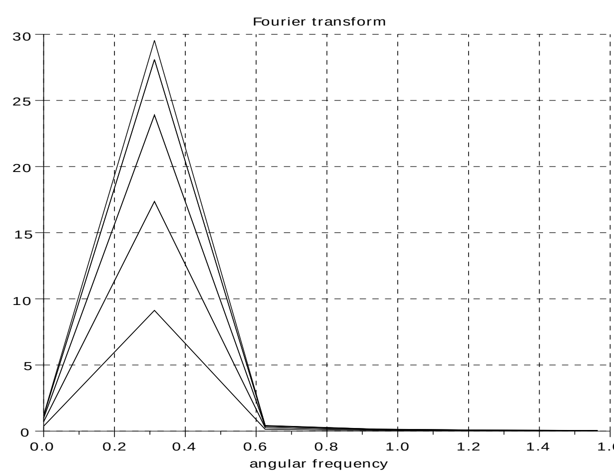

Figure 11: Phase portrait of a system with 9 d.o.f. for $\omega_\epsilon = 0.3128868$

(b) $\int_0^{2\pi} e^{it} S(t,\epsilon)dt = 0, \quad \int_0^{2\pi} e^{-it} S(t,\epsilon)dt = 0$ *uniformly for* ϵ *small enough*

2. *for all* $R > 0$*, there exists* k_R *such that for* $|u| \leq R$ *and* $|v| \leq R$*, the inequality* $|g(t,u,\epsilon) - g(t,v,\epsilon)| \leq k_R|u-v|$ *holds and* $|g(t,0,\epsilon)|$*is bounded; in other words* g *is locally lipschitzian with respect to* u*.*

then, there exists $\gamma > 0$ *such that for* ϵ *small enough,* w_ϵ *is uniformly bounded in* $C^2(0,T_\epsilon)$ *with* $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof. The proof is close to the proof of lemma 6.3 of [JR10]; but it is technically simpler since here we assume g to be locally lipschitzian with respect to u whereas it is only bounded in [JR10].

1. We first consider

$$w_1\text{''} + w_1 = S(t,\epsilon) \tag{146}$$

$$w_1(0) = 0, \quad w_1'(0) = 0 \tag{147}$$

as S is a sum of periodic functions which are uniformly orthogonal to e^{it} and e^{-it}, w_1 is bounded in $\mathcal{C}^2(0,+\infty)$.

2. Then we perform a change of function: $w = w_1 + w_2$, the following equalities hold

$$w_2\text{''} + w_2 = \epsilon g_2(t, w_2, \epsilon) \tag{148}$$

$$w_2(0) = 0, \quad w_2'(0) = 0 \tag{149}$$

with g_2 which satisfies the same hypothesis as g:

for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the following inequality holds $|g_2(t,u,\epsilon) - g_2(t,v,\epsilon)| \leq k_R|u-v|$. Using Duhamel principle, the solution of this equation satisfies:

$$w_2 = \epsilon \int_0^t \sin(t-s) g_2(s, w_2(s), \epsilon)ds \tag{150}$$

from which

$$|w_2(t)| \leq \epsilon \int_0^t |g_2(s, w_2(s), \epsilon) - g_2(s,0,\epsilon)|ds + \epsilon \int_0^t |g_2(s,0,\epsilon)|ds \tag{151}$$

so if $|w| \leq R$, hypothesis of lemma imply

$$|w_2(t)| \leq \epsilon \int_0^t k_R |w_2| ds + \epsilon C t. \tag{152}$$

A corollary of lemma of Bellman-Gronwall, see below, will enable to conclude. It yields

$$|w_2(t)| \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1\right). \tag{153}$$

Now set $T_\epsilon = \sup\{t||w| \leq R\}$, then we have

$$R \leq \frac{C}{k_R}\left(\exp(\epsilon k_R t) - 1\right)$$

this shows that there exists γ such that $|w_2| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0, T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have w in $\mathcal{C}(0, T_\epsilon)$ then as w is solution of (144), it is also bounded in $\mathcal{C}^2(0, T_\epsilon)$.

□

Lemma 5.2. *(Bellman-Gronwall, [bel, Bel64]) Let u, ϵ, β be continuous functions with $\beta \geq 0$,*

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s)u(s)ds \text{ for } 0 \leq t \leq T \tag{154}$$

then

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s)\epsilon(s)\left[\exp(\int_s^t \beta(\tau)d\tau\right] ds \tag{155}$$

Lemma 5.3. *(a consequence of previous lemma, suited for expansions, see [SV85]) Let u be a positive function, $\delta_2 \geq 0$, $\delta_1 > 0$ and*

$$u(t) \leq \delta_2 t + \delta_1 \int_0^t u(s)ds$$

then

$$u(t) \leq \frac{\delta_2}{\delta_1}\left(exp(\delta_1 t) - 1\right)$$

Lemma 5.4. *Let $v_\epsilon = [v_1^\epsilon, \ldots, v_N^\epsilon]^T$ be the solution of the following system:*

$$\omega_1^2 (v_k^\epsilon)\text{"} + \omega_k^2 v_k^\epsilon = S_k(t) + \epsilon g_k(t, v_\epsilon). \tag{156}$$

If ω_1 and ω_k are $\mathbb{Z}$ independent for all $k = 2\ldots N$ and the right hand side satisfies the following conditions with $M > 0$, $C > 0$ prescribed constants:

1. *S_k is a sum of bounded periodic functions, $|S_k(t)| \leq M$ which satisfy the non resonance conditions:*
2. *S_1 is orthogonal to $e^{\pm it}$, i.e. $\int_0^{2\pi} S_1(t)e^{\pm it}dt = 0$ uniformly for ϵ going to zero*
3. *for all $R > 0$ there exists k_R such that for $\|u\| \leq R$, $\|v\| \leq R$, the following inequality holds for $k = 1, \ldots, N$:*

$$|g_k(t, u, \epsilon) - g_k(t, v, \epsilon)| \leq k_R\|u - v\|$$

and $|g_k(t, 0, \epsilon)|$ is bounded *then there exists $\gamma > 0$ such that for ϵ small enough v_ϵ is bounded in $\mathcal{C}^2(0, T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$*

Proof. 1. We first consider the linear system

$$\omega_1^2 (v_{k,1})" + \omega_k^2 v_{k,1} = S_k \tag{157}$$

$$v_{k,1}(0) = 0 \text{ and } (v_{k,1})' = 0 \tag{158}$$

For $k = 1$, with hypothesis 1.a, S_1 is a sum of bounded periodic functions; it is orthogonal to $e^{\pm it}$, there is no resonance. For $k \neq 1$, there is no resonance as $\frac{\omega_k}{\omega_1} \notin \mathbb{Z}$ with hypothesis 1.b.

So $v_{k,1}$ belongs to $C^{(2)}$ for $k = 1, ..., n$.

2. Then we perform a change of function

$$v_k^\epsilon = v_{k,1} + v_{k,2}^\epsilon$$

and $v_{k,2}^\epsilon$ are solutions of the following system :

$$\omega_1^2 (v_{k,2})" + \omega_k^2 v_{k,2} = \epsilon g_{k,2}(t, v_{k,2}, \epsilon), \ k = 1, \dots, N \tag{159}$$

$$v_{k,2}^\epsilon(0) = 0, \ (v_{k,2}^\epsilon)' = 0, \ k = 1, \dots, N \tag{160}$$

with

$$g_{k,2}(t, ..., v_{k,2}^\epsilon, ...) = g_k(t, ..., v_{k,1} + v_{k,2}^\epsilon, ...)$$

where $g_{k,2}$ satisfies the same hypothesis as g_k:
for all $R > 0$ there exists k_R such that for $\| u_k \| \leq R$, $\| v_k \| \leq R$, the following inequality holds for $k = 1, \dots, N$:

$$\| g_{k,2}(t, u_k, \epsilon) - g_{k,2}(t, v_k, \epsilon) \| \leq k_R \| u_k - v_k \| . \tag{161}$$

Using Duhamel principle, the solution or the equation (159) satisfies:

$$v_{k,2}^\epsilon = \epsilon \int_0^t \sin(t-s) g_{k,2}(s, v_{k,2}^\epsilon(s), \epsilon) ds \tag{162}$$

so

$$\| v_{k,2}^\epsilon(t) \| \leq \epsilon \int_0^t \| g_{k,2}(s, v_{k,2}^\epsilon(s), \epsilon) - g_{k,2}(s, 0, \epsilon) \| ds + \epsilon \int_0^t \| g_{k,2}(s, 0, \epsilon) \| ds \tag{163}$$

so with (161), we obtain

$$\| v_{k,2}^\epsilon(t) \| \leq \epsilon \int_0^t k \| v_{k,2}^\epsilon(t) \| ds + \epsilon C t \tag{164}$$

We shall conclude using Bellman-Gronwall lemma; we obtain

$$\| v_{k,2}(t) \| \leq \frac{C}{k_R}(exp(\epsilon k_R t) - 1) \tag{165}$$

this shows that there exists γ such that $|v^{\epsilon}_{k,2}| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0, T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have v_k in $\mathcal{C}(0, T_\epsilon)$ then as v_k is solution of (144), it is also bounded in $\mathcal{C}^2(0, T_\epsilon)$.

□

Theorem 5.1. *(of Poincaré-Lyapunov, for example see [SV85]) Consider the equation*

$$\dot{x} = (A + B(t))x + g(t, x),\ x(t_0) = x_0,\ t \geq t_0$$

where $x, x_0 \in \mathbf{R}^n$, A is a constant matrix $n \times n$ with all its eigenvalues with negative real parts; $B(t)$ is a matrix which is continuous with the property $\lim_{t\to+\infty} \|B(t)\| = 0$. The vector field is continuous with respect to t and x is continuously differentiable with respect to x in a neighborhood of $x = 0$; moreover

$$g(t, x) = o(\|x\|) \text{ when } \|x\| \to 0$$

uniformly in t. Then, there exists constants C, t_0, δ, μ such that if $\|x_0\| < \frac{\delta}{C}$

$$\|x\| \leq C\|x_0\|e^{-\mu(t-t_0)}, t \geq t_0$$

holds

5.2 Numerical computations of Fourier transform

Assuming a function f to be almost-periodic, the fourier coefficients are :

$$\alpha_n = \lim_{T\to+\infty} \int_0^T f(t)e^{-\lambda_n t}dt \tag{166}$$

(for example, see Fourier coefficients of an almost-periodic function in http://www.encyclopediaofmath.org/). For numerical purposes, we chose T large enough and consider the Fourier coefficients of a function of period T equal to f in this interval.

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Chapitre 2

Triple scale analysis of periodic solutions and resonance of some asymmetric non linear vibrating systems

Vu que l'article est assez chargé, nous avons préféré de ne pas y mettre certains résultats numériques et qui seront présentés après l'article. En remarquant l'importance des termes quadratiques et cubiques de la non linéarité, nous avons procédé à quelques expériences où nous avons varié les coefficients c et d.

2.1 Triple scale analysis of periodic solutions and resonance of some asymmetric non linear vibrating systems

Ce travail est réalisé par Nadia Ben Brahim[1] et Bernard Rousselet[2] ; il est déposé dans l'archive ouvert pluridisciplinaire H.A.L.[3] du Centre pour la communication scientique directe et publié dans le Journal of Applied Mathematics and Computing [BR13].

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J. Appl. Math. Comput. manuscript No.
(will be inserted by the editor)

Triple scale analysis of periodic solutions and resonance of some asymmetric non linear vibrating systems

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Abstract We consider *small solutions* of a vibrating mechanical system with smooth non-linearities for which we provide an approximate solution by using a triple scale analysis; a rigorous proof of convergence of the triple scale method is included; for the forced response, a stability result is needed in order to prove convergence in a neighbourhood of a primary resonance. The amplitude of the response with respect to the frequency forcing is described and it is related to the frequency of a free periodic vibration.

Keywords triple scale expansion · periodic solutions · nonlinear vibrations · normal modes · resonance

Mathematics Subject Classification (2000) 34e13 · 34c25 · 70k30 · 74h10 · 74h45

1 Introduction

In this article, we perform a triple scale analysis of small periodic solutions of free vibrations of a discrete structure without damping and with a local smooth non-linearity; then we consider a similar system with damping and a periodic forcing in a resonance situation.

Several experimental studies show that it is possible to detect defects in a structure by considering its vibro-acoustic response to an external actuation; there is a vast literature in applied physics. We recall some papers related to the use of the frequency response for non destructive testing; in particular generation of higher harmonics, cross-modulation of a high frequency by a low frequency (often called intermodulations in telecommunication): [EDK99], [MCG02]; in [DGLV03], "a vibro-acoustic method, based on frequency modulation, is developed in order to detect defects on aluminium and concrete beams"; experiments have been performed on a real bridge by G. Vanderborck with four prestressed cables: two undamaged cables, a damaged one and a safe one but damaged at the anchor. With routine experimental checking of the lowest natural frequency, the presence of the damaged cable had only been found by

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comparison with data collected 15 years ago; the one damaged at the anchor was not found; see details in [LVdb04], [RV05].

However the analysis *per se* of non linear vibration is also an important topic from the academic and industrial viewpoint. In this work, we are interested in the behaviour due to a local non linear stress-strain law; first, we consider free vibration and then forced response of a damped system with excitation frequency close to a frequency of the free system ; so, this local stress-strain law is assumed to be: $N = k\tilde{u} + c\tilde{u}^2 + d\tilde{u}^3$, where N is the normal force and $\tilde{u}$ is the elongation. The elastodynamic problem of continuum mechanics leads after discretization by finite elements to a system of non linear differential equations of second order, thus, this paper deals with such systems with several degrees of freedom. We determine an asymptotic expansion of *small periodic solutions* of a discrete structure; we use the method of triple scale [Nay81] and compare these results with a numerical integration program; also, we perform a numerical Fourier transform to determine the frequencies and compare with that of the linear system.

Our approach is only valid in the low frequency range and we have bypassed the propagation of acoustic waves in the structure; this point has been studied in [JL09],[JL12]. The case of rigid contact which is also important from the point of view of theory and applications has been addressed in several papers, for example [JL01], and a synthesis in [BBL13] ; a numerical method to compute periodic solutions is proposed in [LL11] . Asymptotic expansions have been used for a long time; such methods are introduced in the famous memoir of Poincaré [Poi99]; a classic general book on asymptotic methods is [BM55] with french and English translations [BM62, BM61]; introductory material is in [Nay81], [Mil06]; a detailed account of the averaging method with precise proofs of convergence may be found in [SV85]; an analysis of several methods including multiple scale expansion may be found in [Mur91]; the case of vibrations with unilateral springs have been presented in [JR09, JR10, VLP08, **?**, HR09b, HFR09]; this topic has been presented by H. Hazim at "Congrès Smai" in 2009; more details are to be found in his thesis defended at University of Nice Sophia-Antipolis in 2010. In a forthcoming paper, such a non-smooth case will be considered as well as a numerical algorithm based on the fixed point method used in [Rou11]. The case of vibrations with weak grazing unilateral contact has been presented by S. Junca and Ly Tong at 4th Canadian Conference on Nonlinear Solid Mechanics 2013; in [JPS04]

a numerical approach for large solutions of piecewise linear systems is proposed. A review paper for so called "non linear normal modes" may be found in [KPGV09]; it includes numerous papers published by the mechanical engineering community; several application fields have been addressed by this community; for example in [Mik10] "nonlinear vibro-absorption problem, the cylindrical shell nonlinear dynamics and the vehicle suspension nonlinear dynamics are analysed". Preliminary versions of these results may be found in [BR09] and have been presented in conferences [Bra10, Bra]; a proof of convergence of double scale expansion is to be found in the preliminary work [BR13].

In the present text and in the conclusion, we compare the use of double or triple scale expansion. We emphasize that the use of three time scales, instead of two times scales presented in the preliminary work [BR13], provides a much improved insight in the behavior of the forced response close to resonance. *In this paper*, as an introduction, in a first step, we consider *small solutions* of a system with one degree of freedom; we compare free vibration frequency and the frequency of the periodic forcing for which the amplitude is maximal. Then we address a system with several degrees of freedom, we look for periodic free vibrations (so called non linear normal modes in the mechanical engineering community); we compare this frequency with the response to a periodic forcing close to resonance.

2 One degree of freedom, quadratic and cubic non linearity

We consider a stress-strain law with a strong cubic non linearity:

$$N = k\tilde{u} + \Phi(\tilde{u}, \epsilon) \text{ with } \Phi(\tilde{u}, \epsilon) = mc\tilde{u}^2 + \frac{md}{\epsilon}\tilde{u}^3$$

where ϵ is a small parameter which is also involved in the size of the solution; m is the mass, k the linear rigidity of the spring and $\tilde{u}$ the change of length of the spring; the choice of this scaling provides frequencies which are amplitude dependent at first order.

2.1 Free vibration, triple scale expansion up to second order

Using second Newton law, free vibrations of a mass attached to such a spring are governed by:

$$\ddot{\tilde{u}} + \omega^2\tilde{u} + c\tilde{u}^2 + \frac{d}{\epsilon}\tilde{u}^3 = 0. \tag{1}$$

Remark 1 – We intend to look for a small solution therefore, we consider a change of function $\boxed{\tilde{u} = \epsilon u}$ and obtain the transformed equation:

$$\ddot{u} + \omega^2 u + \epsilon c u^2 + \epsilon d u^3 = 0.$$

In this form, this is a Duffing equation for which exists a vast literature, for example see the expository book [KB2011].

- For the scaling we have chosen, when we use double scale analysis, we remarked in [BR09] that the approximation that we obtain does not involve explicitly the coefficient c of the quadratic term; this coefficient is only involved in the proof of the validity of the expansion. In particular the frequency shift only involves the coefficient d of the cubic term.
- However when we use three time scales, the coefficient of the quadratic term is involved in the frequency shift.
- On the other hand, if we would let $\epsilon \to +\infty$ in (1) , we would get a singular perturbation problem; this is not considered here.

As we look for a *small* solution with a triple scale analysis for time; we set

$$T_0 = \omega t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t, \text{ hence } D_0 u = \frac{\partial u}{\partial T_0}, \quad D_1 u = \frac{\partial u}{\partial T_1} \text{ and } D_2 u = \frac{\partial u}{\partial T_2} \tag{2}$$

and we obtain

$$\begin{aligned} \frac{du}{dt} &= \omega D_0 u + \epsilon D_1 u + \epsilon^2 D_2 u \\ \frac{d^2u}{dt^2} &= \omega^2 D_0^2 u + 2\epsilon\omega D_0 D_1 u + 2\epsilon^2\omega D_0 D_2 u + \epsilon^2 D_1^2 u + 2\epsilon^3 D_1 D_2 u + \epsilon^4 D_2^2 u. \end{aligned}$$

As we look for a small solution we consider initial data $\tilde{u}(0) = \epsilon a + \epsilon^2 v_1 + \mathcal{O}(\epsilon^3)$ and $\dot{\tilde{u}}(0) = \mathcal{O}(\epsilon^3)$; or $u(0) = a + \epsilon v_1 + \mathcal{O}(\epsilon^2)$ and $\dot{u}(0) = \mathcal{O}(\epsilon^2)$; we expand the solution with the *ansatz*

$$u(t) = u(T_0, T_1, T_2) = u^{(1)}(T_0, T_1, T_2) + \epsilon u^{(2)}(T_0, T_1, T_2) + \epsilon^2 r(T_0, T_1, T_2); \tag{3}$$

so we obtain:

$$\frac{du}{dt} = \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2\frac{dr}{dt} = \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2 D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r)$$
$$= [\omega D_0 u^{(1)} + \epsilon D_1 u^{(1)} + \epsilon D_2 u^{(1)}] + \epsilon[\omega D_0 u^{(2)} + \epsilon D_1 u^{(2)} + \epsilon D_2 u^{(2)}]$$
$$+ \epsilon^2[\omega D_0 r + \epsilon D_1 r + \epsilon^2 D_2 r]$$

and with the formula

$$\mathcal{D}_3 r = \frac{1}{\epsilon}\left(\frac{d^2 r}{dt^2} - \omega^2 D_0^2 r\right) = 2\omega D_0 D_1 r + \epsilon\left[2\omega D_0 D_2 r + D_1^2 r + 2\epsilon D_1 D_2 r,\right] + \epsilon^3 D_2^2 r,$$

we get

$$\begin{aligned}\frac{d^2u}{dt^2} &= \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2\frac{d^2r}{dt^2} = \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2 D_0^2 r + \epsilon^3\mathcal{D}_3 r \\ &= \omega^2 D_0^2 u^{(1)} + \epsilon\left[2\omega D_0 D_1 u^{(1)} + \omega^2 D_0^2 u^{(2)}\right] \\ &\quad + \epsilon^2\left[2\omega D_0 D_2 u^{(1)} + D_1^2 u^{(1)} + 2\omega D_0 D_1 u^{(2)} + D_0^2 r\right] \\ &\quad + \epsilon^3\left[2D_1 D_2 u^{(1)} + 2\omega D_0 D_2 u^{(2)} + D_1^2 u^{(2)} + \mathcal{D}_3 r\right] \\ &\quad + \epsilon^4\left[D_2^2 u^{(1)} + 2D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)}\right].\end{aligned} \tag{4}$$

We plug expansions (3),(4) into (1); by identifying the powers of ϵ in the expansion of equation (1), we obtain:

$$\begin{cases} D_0^2 u^{(1)} + u^{(1)} = 0 \\ \omega^2\left[D_0^2 u^{(2)} + u^{(2)}\right] = S_2 \\ \omega^2\left[D_0^2 r + r\right] = S_3 \end{cases} \tag{5}$$

with

$$S_2 = -c{u^{(1)}}^2 - du^{(1)3} - 2\omega D_0 D_1 u^{(1)} \quad \text{and}$$
$$S_3 = -2cu^{(1)}u^{(2)} - 3du^{(1)2}u^{(2)} - 2\omega D_0 D_2 u^{(1)} - D_1^2 u^{(1)} - 2\omega D_0 D_1 u^{(2)} - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)}),$$

with

$$\begin{aligned}R(\epsilon, r, u^{(1)}, u^{(2)}) &= 2D_1 D_2 u^{(1)} + 2\omega D_0 D_2 u^{(2)} + D_1^2 u^{(2)} \\ &\quad + cu^{(2)2} + 2cru^{(1)} + 3du^{(1)}u^{(2)2} + 3du^{(1)2}r + \mathcal{D}_3 r \\ &\quad + \epsilon\left(D_2^2 u^{(1)} + 2D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)}\right) + \epsilon\rho(u^{(1)}, u^{(2)}, r, \epsilon)\end{aligned}$$

and with ρ, a polynomial in r:

$$\begin{aligned}\rho(u^{(1)}, u^{(2)}, r, \epsilon) &= 2cru^{(2)} + du^{(2)3} + +6du^{(1)}u^{(2)}r \\ &\quad + \epsilon(cr^2 + 3du^{(2)2}r + 3du^{(1)}r^2) + \epsilon^2[3du^{(2)}r^2 + \epsilon dr^3].\end{aligned}$$

For convenience, we perform the change of variable $\theta(T_0, T_1, T_2) = T_0 + \beta(T_1, T_2)$; we notice that $D_0\theta = 1; D_1\theta = D_1\beta$ and $D_2\theta = D_2\beta$; we solve the first equation of (5) with $D_0 u^{(1)}(0) = 0$, we get:

$$u^{(1)} = a(T_1, T_2)\ \cos(\theta). \tag{6}$$

Remark 2 We notice that a and β are not constants but functions of time scales T_1 *and* T_2 because u depends on these times scales. The dependence of these functions with respect to T_1 *and* T_2 will be determined by solving the equations of the following orders and eliminating the so-called secular terms.

First, we determine the dependence on T_1; with simple manipulation of the second equation of (5), we obtain

$$S_2 = -\frac{ca^2}{2}(\cos(2\theta)+1) - \frac{da^3}{4}\cos(3\theta) + \cos(\theta)\left(\frac{-3da^3}{4} + 2\omega a D_1\beta\right) + 2\omega D_1 a\sin(\theta)$$

we gather terms at angular frequency ω:

$$S_2 = -\frac{3da^3}{4}cos(\theta) + 2\omega\left[D_1 a\sin(\theta) + aD_1\beta cos(\theta)\right] + S_2^\sharp \quad \text{where}$$

$$S_2^\sharp = \frac{-ca^2}{2}(1+\cos(2\theta)) - \frac{da^3}{4}\cos(3\theta)$$

It appears some terms at the frequency of the system, these terms provide a solution $u^{(2)}$ of the equation (72) which is non periodic and non bounded over long time intervals. We will eliminate these so-called secular terms by imposing:

$$D_1 a = 0 \text{ and } \quad D_1\beta = \frac{3da^2}{8\omega} \tag{7}$$

the solution of the second equation of (5), is:

$$u^{(2)} = \frac{-ca^2}{2\omega^2} + \frac{ca^2}{6\omega^2}\cos(2\theta) + \frac{da^3}{32\omega^2}\cos(3\theta).$$

Remark 3 We have omitted the term at frequency ω which is redundant with $u^{(1)}$; however this choice is connected to the value of the initial condition; see Remark 5.

For the third equation of (5), the unknown is r; this equation includes non linearities; we do not solve it but we show that the solution is bounded on an interval dependent on ϵ. We use the values of $u^{(1)}, u^{(2)}$ in S_3. Intermediate computations:

$$u^{(1)}u^{(2)} = \frac{-5ca^3}{12\omega^2}\cos(\theta) + \frac{ca^3}{12\omega^2}\cos(3\theta) + \frac{da^4}{64\omega^2}(\cos(2\theta)+\cos(4\theta)).$$

$$(u^{(1)})^2u^{(2)} = \frac{-5ca^4}{24\omega^2} + \frac{da^5}{128\omega^2}\cos(\theta) - \frac{ca^4}{6\omega^2}\cos(2\theta) + \frac{da^5}{64\omega^2}\cos(3\theta) + \frac{ca^4}{24\omega^2}\cos(4\theta) + \frac{da^5}{128\omega^2}\cos(5\theta)$$

The right hand side, after some manipulations is:

$$\begin{aligned} S_3 = \sin(\theta)&\left(2\omega D_2 a + 2D_1 a D_1\beta + aD_1^2\beta\right) \\ &+ \cos(\theta)\left(2\omega a D_2\beta - D_1^2 a + a(D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2}\right) \\ &+ S_3^\sharp - \epsilon R(r,\epsilon,u^{(1)},u^{(2)}) \end{aligned}$$

with

$$
\begin{aligned}
S_3^\sharp = \frac{5dca^4}{8\omega^2} + \sin(2\theta)\left(\frac{4ca}{3\omega}D_1 a\right) + \cos(2\theta)\left(\frac{4ca^2}{3\omega}D_1\beta + \frac{15cda^4}{32\omega^2}\right) \\
+ \sin(3\theta)\left(\frac{9da^2}{16\omega}D_1 a\right) + \cos(3\theta)\left(\frac{-c^2a^3}{6\omega^2} - \frac{3d^2a^5}{64\omega^2} + \frac{9da^3}{16\omega}D_1\beta\right) \\
+ \cos(4\theta)(\frac{-5cda^4}{32\omega^2}) + \cos(5\theta)(\frac{-3d^2a^5}{128\omega^2}).
\end{aligned}
$$

By imposing

$$
\begin{aligned}
&2\omega D_2 a + 2D_1 a D_1\beta + aD_1^2\beta = 0 \\
&2\omega a D_2\beta - D_1^2 a + a(D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0
\end{aligned}
$$

we get that $S_3 = S_3^\sharp - \epsilon R(\epsilon, u^{(1)}, u^{(2)}, r)$ no longer contains any term at frequency ω.

As $D_1 a = 0$ and $D_1\beta = \frac{3da^2}{8\omega}$, we obtain

$$
2\omega a D_2\beta + a\left(\frac{9d^2a^4}{64\omega^2}\right) + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0.
$$

So,

$$
D_2 a(T_2) = 0 \quad \textit{and} \quad D_2\beta(T_2) = (-\frac{5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3}). \tag{8}
$$

As a and β do not depend on T_0, we note that:

$$
\begin{cases}
\dfrac{da}{dt} = \epsilon D_1 a + \epsilon^2 D_2 a + \mathcal{O}(\epsilon^3) \\
\dfrac{d\beta}{dt} = \epsilon D_1\beta + \epsilon^2 D_2\beta + \mathcal{O}(\epsilon^3),
\end{cases} \tag{9}
$$

thus taking into account (7) and to (8), we obtain:

$$
\frac{da}{dt} = 0 \qquad \text{and} \qquad \frac{d\beta}{dt} = \epsilon\frac{3da^2}{8\omega} + \epsilon^2(\frac{-5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3}) \tag{10}
$$

therefore, the solution of these equations is:

$$
a = cte \quad \text{and} \quad \beta(t) = \left[\epsilon\frac{3da^2}{8\omega} + \epsilon^2(-\frac{5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3})\right] t. \tag{11}
$$

The constant of integration is chosen to be zero as the initial velocity satisfies $\dot{u}(0) = /(\epsilon^3)$.

In order to show that, r is bounded, after eliminating terms at angular frequency ω, we go back to the t variable in the third equations of (5).

$$
\frac{d^2r}{dt^2} + \omega^2 r = \tilde{S}_3 \tag{12}
$$

with $\tilde{S}_3 = S_3^\sharp(t,\epsilon) - \epsilon\tilde{R}(r,\epsilon,u^{(1)},u^{(2)})$ where

$$S_3^\sharp(t,\epsilon) = \frac{5dca^4}{8\omega^2} + \cos(2(\omega t+\beta(t)))\left(\frac{15cda^4}{32\omega^2} + \frac{cda^4}{2\omega^2}\right) + \sin(2(\omega t+\beta(t)))(\frac{cda^4}{2\omega^2})$$
$$+\cos(3(\omega t+\beta(t)))\left(\frac{-c^2a^3}{6\omega^2} - \frac{3d^2a^5}{64\omega^2} + \frac{27d^2a^5}{128\omega^2}\right) + \sin(3(\omega t+\beta(t)))(\frac{9d^2a^5}{128\omega^2})$$
$$+\cos(4(\omega t+\beta(t)))(\frac{-3cda^4}{32\omega^2}) + \cos(5(\omega t+\beta(t)))(\frac{-3d^2a^5}{128\omega^2})$$

$$\text{and}\quad \tilde{R} = R(\epsilon,r,u^{(1)},u^{(2)}) - \mathcal{D}_3 r.$$

in which the remainder $\tilde{R}$, the functions $u^{(1)}, u^{(2)}$ and their partial derivatives with respect to T_1, T_2 are expressed with the variable t.

Proposition 1 *There exists $\gamma > 0$ such that for all $t \le t_\epsilon = \frac{\gamma}{\epsilon^2}$, the solution $\tilde{u} = \epsilon u$ of (1) has the following expansion,*

$$\begin{cases} \tilde{u}(t) = \epsilon a \ \cos(\nu_\epsilon t) \ + \ \epsilon^2\left(\frac{-ca^2}{2\omega^2} \ + \ \frac{ca^2}{6\omega^2}\cos(2\nu_\epsilon t) + \frac{da^3}{32\omega^2}\cos(3\nu_\epsilon t)\right) + \epsilon^3 r(\epsilon,t) \\ \tilde{u}(0) = \epsilon a + \epsilon^2(\frac{-ca^2}{3\omega^2} + \frac{da^3}{32\omega^2}) + O(\epsilon^3), \dot{u}(0) = O(\epsilon^2) \end{cases} \tag{13}$$

with

$$\nu_\epsilon = \omega + \epsilon\frac{3da^2}{8\omega} + \epsilon^2\left(-\frac{5c^2a^2}{12\omega^3} - \frac{15d^2a^4}{256\omega^3}\right) + \mathcal{O}(\epsilon^3) \tag{14}$$

and r is uniformly bounded in $C^2(0,t_\epsilon)$.

Proof Let us use lemma 1 with equation (12); set $S = S_3^\sharp$; as we have enforced (10), it is a periodic bounded function orthogonal to $e^{\pm it}$, it satisfies lemma hypothesis; similarly set $g = \tilde{R}$; it is a polynomial in variable r with coefficients which are bounded functions, so it is a lipschitzian function on bounded subsets and satisfies lemma hypothesis. □

Remark 4 We notice that if we increase c, there is a change of convexity of the mapping $a \mapsto \nu_\epsilon$; this is an effect which cannot be noticed by just obtaining a first order approximation of the frequency with a double scale approximation of the solution as in [BR13]. See numerical results at the end of subsection 2.3.

Remark 5 We can notice that we can also derive the solution which satisfies $u(0) = \epsilon a$ by adding to the solution $-\epsilon^2(\frac{-ca^2}{3\omega^2} + \frac{da^3}{32\omega^2})\cos(\nu_\epsilon t)$

2.2 Numerical Results

In the figure 1, we find plots of the Fourier transform of solutions; on the left, the linear case, we notice one frequency and on the right, three frequencies in the non linear case. the Fourier transform displays the frequencies, $\nu_1 = 0.164$; $2\nu_1 = 0.329$; $3\nu_1 = 0.493$

We notice good correlation between analytical results of asymptotic expansion and an integration step by step (with Scilab program ODE and numerical fast Fourier transform).

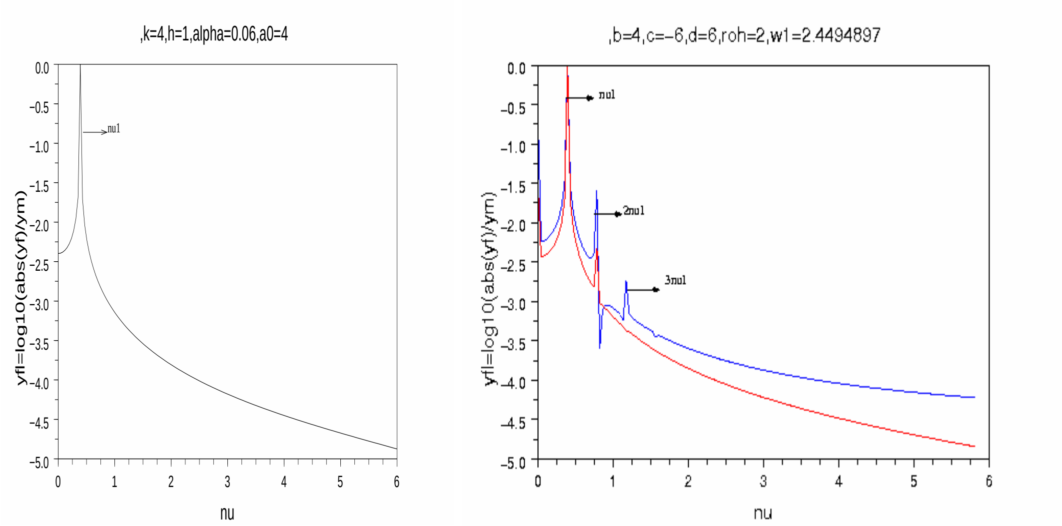

Fig. 1 Dynamic frequency shift(fft) linear(left) and a non linear element with two methods(numerical(blue), asymptotic expansion(red))

2.3 Forced vibration, triple scale expansion up to second order

2.3.1 Derivation of the expansion

Here we consider a similar system with a sinusoidal forcing at a frequency close to the free frequency; in the linear case without damping, it is well known that the solution is no longer bounded when the forcing frequency goes to the free frequency. Here, we consider the mechanical system of previous section but with periodic forcing and we include some damping term; the scaling of the forcing term is chosen so that the expansion works properly; this is a known point, for example see [Nay86].

$$\ddot{\tilde{u}} + \omega^2 \tilde{u} + \epsilon \lambda \dot{\tilde{u}} + c\tilde{u}^2 + \frac{d}{\epsilon}\tilde{u}^3 = \epsilon^2 F_m \cos(\tilde{\omega_\epsilon} t), \tag{15}$$

where $F_m = \frac{F}{m}$ with the mass m; we assume positive damping, $\lambda > 0$ and excitation frequency ω is *close* to an eigenfrequency of the linear system in the following way:

$$\tilde{\omega_\epsilon} = \omega + \epsilon\sigma. \tag{16}$$

Remark 6 – We look for a small solution with a triple scale expansion; as for the free vibrations, we consider a change of function $\boxed{\tilde{u} = \epsilon u}$ and obtain the transformed equation

$$\ddot{u} + \omega^2 u + \epsilon\lambda\dot{u} + \epsilon c u^2 + \epsilon d u^3 = \epsilon F_m \cos(\tilde{\omega_\epsilon} t).$$

– To simplify the computations, the fast scale T_0 is chosen to be ϵ dependent.

We set:

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t \text{ and } T_2 = \epsilon^2 t, \text{ therefore } D_0 u = \frac{\partial u}{\partial T_0}, \quad D_1 u = \frac{\partial u}{\partial T_1} \text{ and } D_2 u = \frac{\partial u}{\partial T_2},$$

so

$$\begin{aligned}
&\frac{du}{dt} = \tilde{\omega}_\epsilon D_0 u + \epsilon D_1 u + \epsilon^2 D_2 u \quad \text{and} \\
&\frac{d^2u}{dt^2} = \tilde{\omega}_\epsilon^2 D_0^2 u + 2\epsilon\tilde{\omega}_\epsilon D_0 D_1 u + 2\epsilon^2 D_0 D_2 u + \epsilon^2 D_1^2 u + 2\epsilon^3 D_1 D_2 u + \epsilon^4 D_2^2 u.
\end{aligned} \tag{17}$$

With (16), (17) and the following *ansatz*, we look for a small solution:

$$u(t) = u(T_0, T_1, T_2) = u^{(1)}(T_0, T_1, T_2) + \epsilon u^{(2)}(T_0, T_1, T_2) + \epsilon^2 r(T_0, T_1, T_2) \tag{18}$$

we obtain:

$$\begin{aligned}
\frac{du}{dt} &= \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2\frac{dr}{dt} = \frac{du^{(1)}}{dt} + \epsilon\frac{du^{(2)}}{dt} + \epsilon^2 D_0 r + \epsilon^2(\frac{dr}{dt} - D_0 r) \\
&= [(\omega + \epsilon\sigma)D_0 u^{(1)} + \epsilon D_1 u^{(1)} + \epsilon D_2 u^{(1)}] + \epsilon[(\omega + \epsilon\sigma)D_0 u^{(2)} + \epsilon D_1 u^{(2)} + \epsilon^2 D_2 u^{(2)}] \\
&\qquad + \epsilon^2\omega D_0 r + \epsilon^2(\frac{dr}{dt} - \omega D_0 r)
\end{aligned}$$

where we remark that $\frac{dr}{dt} - \omega D_0 r = \epsilon D_1 r + \epsilon^2 D_2 r$ is of degree 1 in ϵ. For the second derivative, as for the case without forcing, we introduce

$$\begin{aligned}
\mathcal{D}_3 r &= \frac{1}{\epsilon}(\frac{d^2 r}{dt^2} - \omega^2 D_0^2 r) \\
&= 2\tilde{\omega} D_0 D_1 r + \epsilon\left[2\tilde{\omega} D_0 D_2 r + D_1^2 r + 2\epsilon D_2 D_1 r\right] + \epsilon^3 D_2^2 r
\end{aligned}$$

and we get

$$\begin{aligned}
\frac{d^2u}{dt^2} &= \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2\frac{d^2r}{dt^2} = \frac{d^2u^{(1)}}{dt^2} + \epsilon\frac{d^2u^{(2)}}{dt^2} + \epsilon^2\tilde{\omega}^2 D_0^2 r + \epsilon^3\mathcal{D}_3 r \\
&= \tilde{\omega}^2 D_0^2 u^{(1)} + \epsilon\left[2\tilde{\omega} D_0 D_1 u^{(1)} + \tilde{\omega}^2 D_0^2 u^{(2)}\right] \\
&\quad + \epsilon^2\left[2\tilde{\omega} D_0 D_2 u^{(1)} + D_1^2 u^{(1)} + 2\tilde{\omega} D_0 D_1 u^{(2)} + \tilde{\omega}^2 D_0^2 r\right] \\
&\quad + \epsilon^3\left[2 D_1 D_2 u^{(1)} + 2\tilde{\omega} D_0 D_2 u^{(2)} + D_1^2 u^{(2)} + \mathcal{D}_3 r\right] \\
&\quad + \epsilon^4\left[D_2^2 u^{(1)} + 2 D_1 D_2 u^{(2)} + \epsilon D_2^2 u^{(2)}\right].
\end{aligned}$$

We plug previous expansions into (15); we obtain:

$$\begin{cases}
D_0^2 u^{(1)} + u^{(1)} = 0 \\
\omega^2\left(D_0^2 u^{(2)} + u^{(2)}\right) = S_2 \\
\omega^2\left(D_0^2 r + r\right) = S_3
\end{cases} \tag{19}$$

with

$$S_2 = -c{u^{(1)}}^2 - du^{(1)3} - 2\omega D_0 D_1 u^{(1)} - \lambda\omega D_0 u^{(1)} - 2\omega\sigma D_0^2 u^{(1)} + F_m\cos(T_0) \quad \text{and} \tag{20}$$

$$S_3 = -2cu^{(1)}u^{(2)} - 3du^{(1)2}u^{(2)} - 2\omega D_0 D_2 u^{(1)} - D_1^2 u^{(1)} - 2\omega D_0 D_1 u^{(2)} - \sigma^2 D_0^2 u^{(1)} - 2\sigma D_0 D_1 u^{(1)} \tag{21}$$

$$- 2\omega\sigma D_0^2 u^{(2)} - \lambda\omega D_0 u^{(2)} - \lambda D_1 u^{(1)} - \lambda\sigma D_0 u^{(1)} - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)}) \tag{22}$$

with

revoir

$$R(\epsilon, r, u^{(1)}, u^{(2)}) = 2D_1D_2u^{(1)} + 2\omega D_0D_2u^{(2)} + D_1^2u^{(2)}$$
$$+ cu^{(2)2} + 2cu^{(1)}r + 3du^{(1)}u^{(2)2} + 3du^{(1)2}r + \lambda(\omega D_0r + D_2u^{(1)} + D_1u^{(2)} + \epsilon D_2u^{(2)})$$
$$+ \epsilon\left(D_2^2u^{(1)} + 2D_1D_2u^{(2)} + \epsilon D_2^2u^{(2)}\right) + \mathcal{D}_3r + \lambda(\frac{dr}{dt} - \omega D_0r) + \epsilon\rho(u^{(1)}, u^{(2)}, r, \epsilon)$$

and

$$\rho(u^{(1)}, u^{(2)}, r, \epsilon) = 2cru^{(2)} + du^{(2)3} + +6du^{(1)}u^{(2)}r$$
$$+ \epsilon(cr^2 + 3du^{(2)2}r + 3du^{(1)}r^2) + \epsilon^2[3du^{(2)}r^2 + \epsilon dr^3].$$

We solve the first equation of (19):

$$u^{(1)} = a(T_1, T_2)\ \cos\theta \tag{23}$$

where we have set $\theta(T_0, T_1, T_2) = T_0 + \beta(T_1, T_2)$; we use $\cos(T_0) = \cos(\theta)\cos(\beta) + \sin(\theta)\sin(\beta)$ and we obtain

$$S_2 = -\frac{ca^2}{2}(\cos(2\theta) + 1) - \frac{da^3}{4}\cos(3\theta) + \sin(\theta)\left[2\omega D_1a + \lambda\omega a + F_m\sin(\beta)\right]$$
$$+ \cos(\theta)\left[2\omega aD_1\beta - \frac{3da^3}{4} + 2\omega a\sigma + F_m\cos(\beta)\right]$$

or $S_2 = \cos(\theta)\left[\frac{-3da^3}{4} + F_m\cos(\beta)\right] + 2\omega[D_1a\sin(\theta) + a(D_1\beta + \sigma)\cos(\theta)]$
$$+ \sin(\theta)\left[\lambda\omega a + F_m\sin(\beta)\right] + S_2^\sharp$$

with $S_2^\sharp = -\frac{ca^2}{2}(\cos(2\theta) + 1) - \frac{da^3}{4}\cos(3\theta).$

By imposing

$$\begin{cases} 2\omega D_1a + \lambda\omega a = -F_m\sin(\beta) \\ 2\omega aD_1\beta + 2\omega a\sigma - \frac{3da^3}{4} = -F_m\cos(\beta), \end{cases} \tag{24}$$

the solution of the second equation of (19) is:

$$u^{(2)} = \frac{-ca^2}{2\omega^2}\ +\ \frac{ca^2}{6\omega^2}\cos(2\theta) + \frac{da^3}{32\omega^2}\cos(3\theta) \tag{25}$$

where we have omitted the term at the frequency ω is which redundant with $u^{(1)}$.

The third equation of (19) includes non linearities, the unknown is r, we do not solve it, but we show that the solution is bounded on an interval which is ϵ dependent; the right hand side is:

$$S_3 = \sin\theta\left[2\omega D_2a + \lambda aD_1\beta + 2D_1aD_1\beta + aD_1^2\beta + 2\sigma D_1a + \lambda a\sigma\right]$$
$$+ \cos\theta\left[2\omega aD_2\beta - \lambda D_1a - D_1^2a + a(D_1\beta)^2 + \sigma^2a + 2\sigma aD_1\beta + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2}\right]$$
$$+ S_3^\sharp - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)})$$

where revoir

$$S_3^\sharp = \frac{5cda^4}{8\omega^2} + \sin 2\theta \left[\frac{4ca}{3\omega} D_1 a + \lambda \frac{ca^2}{3\omega}\right] + \cos 2\theta \left[\frac{4ca^2}{3\omega} D_1\beta + \frac{15cda^4}{32\omega^2}\right] + \sin 3\theta \left[\frac{9da^2}{16\omega} D_1 a + \frac{3\lambda da^3}{16\omega}\right] + \cos 3\theta \left[\frac{9da^3}{16\omega} D_1\beta - \frac{c^2a^3}{6\omega^2} - \frac{3d^2a^5}{64\omega^2}\right] + \cos 4\theta \left[\frac{-3cda^4}{32\omega^2}\right] - \frac{3d^2a^5}{128\omega^2}\cos 5\theta \quad (26)$$

To eliminate the secular terms, we impose:

$$\begin{cases} 2\omega D_2 a + \lambda a D_1\beta + 2D_1 a D_1\beta + a D_1^2\beta + 2\sigma D_1 a + \lambda a\sigma = 0 \\ 2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(D_1\beta)^2 + \sigma^2 a + 2\sigma a D_1\beta + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0. \end{cases} \quad (27)$$

In the system (24) the expression of $D_1 a, D_1\beta$ can be extracted:

$$\begin{cases} D_1 a = -\frac{F_m \sin(\beta)}{2\omega} - \frac{\lambda a}{2} \\ D_1\beta = -\sigma - \frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega} \end{cases} \quad (28)$$

As the functions a and β do not depend on T_0, the following relations hold:

$$\frac{da}{dt} = \epsilon D_1 a + \epsilon^2 D_2 a + /\epsilon^3) \quad (29)$$

$$\frac{d\beta}{dt} = \epsilon D_1\beta + \epsilon^2 D_2\beta + /\epsilon^3). \quad (30)$$

We are going to express $\frac{da}{dt}, \frac{d\beta}{dt}$ as functions of a, β. We manipulate equation (27)

$$\begin{cases} 2\omega D_2 a + (\lambda a + 2D_1 a)(\sigma + D_1\beta) - a D_1^2\beta = 0 \\ 2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(\sigma + D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0 \end{cases}$$

then, we replace $D_1 a, D_1\beta$ by their expression in (28), we get

$$\begin{cases} 2\omega D_2 a - \frac{F_m\sin(\beta)}{\omega}(\sigma + D_1\beta) - a D_1^2\beta = 0 \\ -2\omega a D_2\beta - \lambda D_1 a - D_1^2 a + a(\sigma + D_1\beta)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0 \end{cases}$$

and

$$\begin{cases} 2\omega D_2 a - \frac{F_m\sin(\beta)}{\omega}(-\frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}) - a D_1^2\beta = 0 \\ -2\omega a D_2\beta - \lambda(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) - D_1^2 a + a(\frac{F_m\cos(\beta)}{2a\omega} - \frac{3da^2}{8\omega})^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega 2} = 0. \end{cases} \quad (31)$$

On the other hand, we can determine $D_1^2 a$ and $D_1^2\beta$ by differentiating (28);

$$D_1^2 a = -\frac{F_m\cos(\beta) D_1\beta}{2\omega} - \frac{\lambda D_1 a}{2}$$

$$D_1^2\beta = \frac{F_m\sin(\beta) D_1\beta}{2a\omega} + \left(\frac{F_m\cos(\beta)}{2a^2\omega} + \frac{3da}{4\omega}\right) D_1 a$$

or with (28)

$$D_1^2 a = -\frac{F_m\cos(\beta)}{2\omega}\left(-\sigma - \frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}\right) - \frac{\lambda}{2}\left(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}\right)$$
$$D_1^2\beta = \frac{F_m\sin(\beta)}{2a\omega}\left(-\sigma - \frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}\right) + \left(\frac{F_m\cos(\beta)}{2a^2\omega} + \frac{3da}{4\omega}\right)\left(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}\right)$$

or

$$D_1^2 a = \frac{\sigma F_m\cos(\beta)}{2\omega} + \frac{F_m^2\cos^2(\beta)}{4a\omega^2} - \frac{3da^2F_m\cos(\beta)}{16\omega^2} + \frac{\lambda F_m\sin(\beta)}{4\omega} + \frac{\lambda^2 a}{4}$$
$$D_1^2\beta = -\frac{\sigma F_m\sin(\beta)}{2a\omega} - \frac{F_m^2\sin(\beta)\cos(\beta)}{2a^2\omega^2} - \frac{3daF_m\sin(\beta)}{16\omega^2} - \frac{\lambda F_m\cos(\beta)}{4a\omega} - \frac{3d\lambda a^2}{8\omega}.$$

Then, in (31) we use previous formula

$$\begin{cases} 2\omega D_2 a - \frac{F_m\sin(\beta)}{\omega}\left(-\frac{F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}\right) \\ \qquad + a\left(-\frac{\sigma F_m\sin(\beta)}{2a\omega} - \frac{F_m^2\sin(\beta)\cos(\beta)}{2a^2\omega^2} - \frac{3daF_m\sin(\beta)}{16\omega^2} - \frac{\lambda F_m\cos(\beta)}{4a\omega} - \frac{3d\lambda a^2}{8\omega}\right) = 0 \\ 2\omega a D_2\beta - \lambda\left(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}\right) - \left(\frac{\sigma F_m\cos(\beta)}{2\omega} + \frac{F_m^2\cos^2(\beta)}{4a\omega^2} - \frac{3da^2F_m\cos(\beta)}{16\omega^2} + \frac{\lambda F_m\sin(\beta)}{4\omega} + \frac{\lambda^2 a}{4}\right) \\ \qquad + a\left(\frac{-F_m\cos(\beta)}{2a\omega} + \frac{3da^2}{8\omega}\right)^2 + \frac{5c^2a^3}{6\omega^2} - \frac{3d^2a^5}{128\omega^2} = 0 \end{cases}$$

we manipulate

$$\begin{cases} 2\omega D_2 a - \frac{9da^2F_m\sin(\beta)}{16\omega^2} - \frac{\sigma F_m\sin(\beta)}{2\omega} - \frac{\lambda F_m\cos(\beta)}{4\omega} - \frac{3d\lambda a^3}{8\omega} = 0 \\ 2\omega a D_2\beta + \frac{\lambda F_m\sin(\beta)}{4\omega} + \frac{\lambda^2 a}{4} - \frac{\sigma F_m\cos(\beta)}{2\omega} - \frac{3da^2F_m\cos(\beta)}{16\omega^2} - \frac{15d^2a^5}{128\omega^2} + \frac{5c^2a^3}{6\omega^2} = 0 \end{cases}$$

and we obtain:

$$\begin{cases} D_2 a = \frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m\sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da^2F_m\sin\beta}{32\omega^3} \\ D_2\beta = -\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2 a} + \frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2 a}. \end{cases} \tag{32}$$

Now we return to (29) introducing (28) and (32), we obtain:

$$\begin{cases} \frac{da}{dt} = \epsilon\left(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}\right) \\ \qquad + \epsilon^2\left(\frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m\sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da^2F_m\sin\beta}{32\omega^3}\right) + O(\epsilon^3) \\ \frac{d\beta}{dt} = \epsilon\left(-\sigma + \frac{3da^2}{8\omega} - \frac{F_m\cos(\beta)}{2a\omega}\right) + \epsilon^2\left(-\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a^2}{12\omega^3}\right. \\ \qquad \left. + \frac{\sigma F_m\cos\beta}{4\omega^2 a} + \frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2 a}\right) + O(\epsilon^3) \end{cases} \tag{33}$$

*Orientation: amplitude and phase equation.*Equations (33) ensure that $S_3^\sharp$ has no term at frequency of ω_1 or which goes to ω_1 .
This will allow us to justify this expansion in certain conditions; before we need to consider the stationnary solution of the system (33) and the stability of the solution close to the stationary solution. This equation (33) is an extension for triple scale analysis of a similar equation introduced in a preliminary work with double scale analysis in [BR13].

Remark 7 In this approach, we are using *the method of reconstitution*; this term has been introduced in 1985 in [Nay86] in order to resolve a discrepancy between higher order approximation solutions obtained by multi scales method on the one hand and generalised averaging method on the other hand; it has been discussed in [VLP08] and from the engineering point of view, the controversy has been resolved in [Nay05]; however the present mathematical proof of convergence seems new.

Remark 8 The previous equations are of importance to derive the solution of the equation (1); their stationary solution will provide an approximate periodic solution of (1).

2.3.2 Stationnary solution and stability

Let us consider the stationary solution of(33), it satisfies:

$$\begin{cases} g_1(a,\beta,\sigma,\epsilon) = 0, \\ g_2(a,\beta,\sigma,\epsilon) = 0 \end{cases} \tag{34}$$

with

$$\begin{cases} g_1 = \epsilon(-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) + \\ \qquad \epsilon^2(\frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m \sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da^2F_m\sin\beta}{32\omega^3}) + \mathcal{O}(\epsilon^3) \\ g_2 = \epsilon(-\sigma + \frac{3da^2}{8\omega} - \frac{F_m\cos(\beta)}{2a\omega}) \\ \qquad +\epsilon^2(-\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2 a} + \frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2 a}) + \mathcal{O}(\epsilon^3). \end{cases} \tag{35}$$

Now, we study the stability of the solution of (35) in a neighbourhood of this stationary solution noted $(\bar{a},\bar{\beta})$; set $a = \bar{a} + \tilde{a}$ and $\beta = \bar{\beta} + \tilde{\beta}$, the linearised system is written :

$$\begin{pmatrix} \frac{d\tilde{a}}{dt} \\ \frac{d\tilde{\beta}}{dt} \end{pmatrix} = J \begin{pmatrix} \tilde{a} \\ \tilde{\beta} \end{pmatrix}$$

with the jacobian matrix

$$J = \begin{pmatrix} \partial_{\bar{a}} g_1 & \partial_\beta g_1 \\ \partial_{\bar{a}} g_2 & \partial_\beta g_2 \end{pmatrix}$$

we compute the partial derivatives:

$$\partial_{\bar{a}} g_1 = \epsilon(-\frac{\lambda}{2}) + \mathcal{O}(\epsilon^2) \qquad \partial_{\bar{a}} g_2 = \epsilon\left(\frac{3d\bar{a}}{4\omega} + \frac{F_m\cos(\beta)}{2a^2\omega}\right) + \mathcal{O}(\epsilon^2)$$

$$\partial_\beta g_1 = -\epsilon\frac{F_m\cos(\beta)}{2\omega} + \mathcal{O}(\epsilon^2) \qquad \partial_\beta g_{12} = \epsilon\frac{F_m\sin(\beta)}{2a\omega} + \mathcal{O}(\epsilon^2)$$

or:

$$\partial_{\bar{a}} g_1 = \epsilon(-\frac{\lambda}{2}) + \mathcal{O}(\epsilon^2) \qquad \partial_{\bar{a}} g_2 = \epsilon(\frac{\sigma}{\bar{a}} + \frac{9d\bar{a}}{8\omega}) + \mathcal{O}(\epsilon^2)$$

$$\partial_\gamma g_1 = \epsilon(\sigma\bar{a} - \frac{3d\bar{a}^3}{8\omega}) + \mathcal{O}(\epsilon^2) \qquad \partial_\gamma g_2 = \epsilon(-\frac{\lambda}{2}) + \mathcal{O}(\epsilon^2)$$

The matrix trace is $tr(J) = -\lambda\epsilon$ and the determinant is

$$\det(J) = \epsilon^2\left[-\frac{\lambda^2}{4} + \sigma^2 - \frac{3d\sigma\bar{a}^2}{2\omega} + \frac{27d^2\bar{a}^4}{64\omega^2}\right] + O(\epsilon^3) \tag{36}$$

the two eigenvalues are negative for ϵ is small enough; when

$$\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$$

then the solution of the linearised system goes to zero; with the theorem of Poincaré-Lyapunov (look in the appendix for the theorem 1) when the initial data is close enough to the stationary solution, the solution of the system (33), goes to the stationary solution.

Proposition 2 *When*

$$\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$$

and ϵ small enough, the stationary solution $(\bar{a}, \bar{\beta})$ of (33) is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary solution of (35), it remains close to it and converges to it); to the stationary case corresponds the approximate solution $\tilde{u}_{app} = \epsilon u_{app}$ of (15)

$$\tilde{u}_{app} = \epsilon\bar{a}\cos(\tilde{\omega}_\epsilon t + \bar{\beta}) + \epsilon^2\left[\frac{-c\bar{a}^2}{2\omega^2} + \frac{c\bar{a}^2}{6\omega^2}\cos(2(\tilde{\omega}_\epsilon t + \bar{\beta})) + \frac{d\bar{a}^3}{32\omega^2}\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}))\right]$$

with

$$\tilde{\omega}_\epsilon = \omega + \epsilon\sigma$$

It is periodic up to the order two.

Remark 9 The expression of u_{app} uses the remark

$$u^{(1)} = a\ \cos(T_0 + \beta) = a\ \cos(\tilde{\omega}_\epsilon t + \beta)$$

and similarly for $u^{(2)}$.

With this result of stability, we can state precisely the approximation of the solution of (15)

2.3.3 Convergence of the expansion

Proposition 3 *Consider the solution $\tilde{u} = \epsilon u$ of (15) with initial conditions*

$$\tilde{u}(0) = \epsilon a_0\cos(\beta_0) + \epsilon^2[\frac{-ca_0^2}{2\omega^2} + \frac{ca_0^2}{6\omega^2}\cos(2\beta_0) + \frac{da^3}{32\omega^2}]\cos(3\beta_0) + \mathcal{O}(\epsilon^3), \tag{37}$$

$$\dot{\tilde{u}}(0) = -\epsilon\omega a_0\sin(\beta_0) + \epsilon^2[\frac{-ca_0^2}{2\omega^2}\sin(2\beta_0) - \frac{da_0^3}{32\omega^2}\sin(3\beta_0)] + \mathcal{O}(\epsilon^3) \tag{38}$$

with (a_0, β_0) close of the stationary solution $(\bar{a}, \bar{\beta})$;

$$|a_0 - \bar{a}| \leq \epsilon^2 C_1, |\beta - \bar{\beta}| \leq \epsilon^2 C_1$$

when $\sigma \leq \frac{3d\bar{a}^2}{4\omega} - \frac{1}{2}\sqrt{\frac{9d^2\bar{a}^4}{16\omega^2} - \lambda^2}$ and ϵ small enough, there exists $\varsigma > 0$ such that for all $t < t_\epsilon = \frac{\varsigma}{\epsilon^2}$, the following expansion of $\tilde{u} = \epsilon u$ is satisfied

$$\left\{\begin{array}{l} \tilde{u}(t) = \epsilon a(t)\cos(\tilde{\omega}_\epsilon t + \beta(t)) + \\ \qquad \epsilon^2[\frac{-ca^2}{2\omega^2} + \frac{ca^2}{6\omega^2}\cos(2(\tilde{\omega}t + \beta(t))) + \frac{da^3}{32\omega^2}\cos(3(\tilde{\omega}t + \beta(t)))] + \epsilon^3 r(\epsilon, t) \end{array}\right. \tag{39}$$

with $\tilde{\omega}_\epsilon = \omega + \epsilon\sigma$ and r uniformly bounded in $C^2(0, t_\epsilon)$ and with $a,\ \beta$ solution of (33)

Proof Indeed after eliminating terms at frequency ν_1, we go back to the variable t for the third equation (19).

$$\frac{d^2r}{dt^2}+\omega^2 r=\tilde{S}_3$$

with
$\tilde{S_3}=S_3^\sharp(t,\epsilon)-\epsilon\tilde{R}(u^{(1)},u^{(2)},r,\epsilon)$ with $\tilde{R}=R-\mathcal{D}_3 r-\lambda(\frac{dr}{dt}-D_0 r)$
with all the terms expressed with the variable t. We express $S_2^\sharp$ in (26) by inserting $D_1 a, D_1\beta$ by their expressions in (24) and using $\theta=\tilde{\omega}_\epsilon t+\beta$; this function is not periodic but is *close* to a periodic function $S_3^\sharp$ by replacing β by $\bar{\beta}$.

As the solution of (33) is stable, for $t\leq\frac{\varsigma}{\epsilon^2}$:

$$|\beta(\epsilon t,\epsilon^2 t)-\bar{\beta}|\leq\epsilon^2 C_1,\quad |a(\epsilon t,\epsilon^2 t)-\bar{a}|\leq\epsilon^2 C_2$$

and

$$|S_3^\sharp-S_3^\natural|\leq\epsilon^2 C_3$$

so this difference may be included in the remainder $\tilde{R}$. We use lemma 5.1 of Appendix (already introduced in [BR13]); with $S=S_3^\natural$; it satisfies lemma hypothesis; similarly, we use $R=\tilde{R}$; it satisfies the hypothesis because it is a polynomial in the variables r,u_1,ϵ,with coefficients which are bounded functions, so it is lipschitzian on bounded subsets.

□

Remark 10 The previous proposition states that for well prepared data close to the stationary solution, the triple scales approximation converges in the sense that the difference between the solution and its approximation is equal to $\epsilon^3 r$ where r is a function which remains bounded in $C^2(0,t_\epsilon)$ with $t_\epsilon=\frac{\gamma}{\epsilon}$, for some constant γ, with ϵ going to 0.

2.3.4 Maximum of the stationary solution, primary resonance

We consider the stationary solution of (33), it satisfies,

$$\begin{cases} g_1(a,\beta,\sigma,\epsilon)=0,\\ g_2(a,\beta,\sigma,\epsilon)=0\end{cases}\tag{40}$$

with formulae (35). We are going to find an expansion of a,β,σ with respect to the small parameter ϵ when $\sigma\mapsto a$ reaches a maximum. The idea is that the functions $(\sigma,\epsilon)\mapsto(a,\beta)$ are defined implicitly by the previous equations; the jacobian matrix is

$$\begin{pmatrix} g_{1a}, & g_{1\beta} & g_{1\sigma} & g_{1\epsilon}\\ g_{2a} & g_{2\beta} & g_{2\sigma} & g_{2\epsilon}\end{pmatrix}$$

and its sub matrix $J_{a\beta}$ is:

$$J(a,\beta)=\begin{pmatrix} g_{1a} & g_{1\beta}\\ g_{2a} & g_{2\beta}\end{pmatrix}$$

in paragraph 2.3.2, we have proved previously that when σ,ϵ are small enough, $J_{a\beta}\neq 0$ and so with the implicit function theorem, in a neighbourhood of the stationary solution, there exists a regular function

$$(\sigma,\epsilon)\longmapsto(a,\beta).$$

We first transform (34) (35) in the following way

$$g_1(a,\beta,\sigma,\epsilon) = (-\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) + \quad \epsilon A_1(a,\beta,\sigma) + \mathcal{O}(\epsilon^2) = 0 \tag{41}$$

$$g_2(a,\beta,\sigma,\epsilon) = (-\sigma - \frac{3da^2}{8\omega} - \frac{F_m\cos(\beta)}{2a\omega}) + \quad \epsilon A_2(a,\beta,\sigma) + \mathcal{O}(\epsilon^2) = 0 \tag{42}$$

with

$$A_1(a,\beta,\sigma) = \frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m\sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da^2F_m\sin\beta}{32\omega^3}$$

$$A_2(a,\beta,\sigma) = -\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2 a} + \frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2 a}$$

We derive a first approximation of $\sin\beta$ and $\cos\beta$ by neglecting terms of order one in ϵ:

$$\begin{cases} \frac{F_m\sin\beta}{2\omega} = -\frac{\lambda a}{2} + \mathcal{O}(\epsilon) \\ \frac{F_m\cos\beta}{2\omega} = \frac{3da^3}{8\omega} - \sigma a + \mathcal{O}(\epsilon) \end{cases} \tag{43}$$

Using $\frac{dg_1}{d\sigma} = 0$, we get

$$\frac{F_m\cos(\beta)}{2\omega}\frac{\partial\beta}{\partial\sigma} - \frac{\lambda}{2}\frac{\partial a}{\partial\sigma} + \epsilon\frac{dA_1}{d\sigma} + \mathcal{O}(\epsilon) = 0 \tag{44}$$

When a is maximum with respect to σ, we get another equation $\frac{\partial a}{\partial\sigma} = 0$; with previous equation, we get a third equation $g_3 = 0$ with

$$g_3(a,\beta,\sigma,\epsilon) = \frac{F_m\cos(\beta)}{2\omega}\frac{\partial\beta}{\partial\sigma} + \epsilon\frac{dA_1}{d\sigma} + \mathcal{O}(\epsilon)$$

We have for $\epsilon = 0$, $\frac{\partial g_3}{\partial a} = 0$, $\frac{\partial g_3}{\partial\sigma} = 0$; we denote $a_0^*, \beta_0^*, \sigma_0^*$ the solution of the 3 equations for $\epsilon = 0$.

We differentiate (43) with respect to σ; when $\frac{\partial a}{\partial\sigma} = 0$, we obtain for the first approximation

$$\begin{cases} \frac{F_m\cos(\beta_0^*)}{\omega}\frac{\partial\beta_0^*}{\partial\sigma} = 0, \\ -\frac{F_m\sin(\beta_0^*)}{2\omega}\frac{\partial\beta_0^*}{\partial\sigma} + a_0^* = 0 \end{cases} \tag{45}$$

and so $\cos(\beta_0^*) = 0$, $\sin(\beta_0^*) = \pm 1$; if we use (41), we notice that a change of sign of $\sin(\beta_0^*)$ changes the sign of a; so we choose $\sin(\beta_0^*) = -1$ and a_0 has the sign of F_m; then with (41), (42), the following equalities hold:

$$a_0^* = \frac{F_m}{\lambda\omega},\ \sigma_0^* = \frac{3da_0^{*2}}{8\omega} = \frac{3dF_m^2}{8\lambda^2\omega^3}; \tag{46}$$

with (45), we get also $\frac{\partial\beta_0^*}{\partial\sigma} = \frac{2\omega a_0^*}{F_m} = \frac{2}{\lambda}$. We remark that c is not involved in these formulas. Then we can compute for $\epsilon = 0$, $\frac{\partial g_3}{\partial a} = 0$; $\frac{\partial g_3}{\partial\beta} = -\frac{F_m\sin(\beta)}{2\omega}\frac{\partial\beta}{\partial\sigma} = -\frac{F_m}{\lambda\omega}$; $\frac{\partial g_3}{\partial\sigma} = 0$. So we obtain that the determinant of the extended matrix

$$J^{\clubsuit}(a,\beta,\sigma) = \begin{pmatrix} g_{1a} & g_{1\beta} & g_{1,\sigma} \\ g_{2a} & g_{2\beta} & g_{2,\sigma} \\ g_{3a} & g_{3\beta} & g_{3,\sigma} \end{pmatrix}$$

is not zero for $(a_0^*, \beta_0^*, \sigma_0^*)$; so once more, we can use the implicit function theorem to define differentiable functions

$$\epsilon \longmapsto (a^*, \beta^*, \sigma^*)$$

where we denote a^*, β^*, σ^* the solution of the 3 equations.

After this first approximation, we look for an expansion of these functions: $\epsilon \longmapsto (a^*, \beta^*, \sigma^*)$;

$$a^* = a_0^* + \epsilon a_1^* + \mathcal{O}(\epsilon^2),\ \beta^* = \beta_0^* + \epsilon\beta_1^* + \mathcal{O}(\epsilon^2),\ \sigma^* = \sigma_0^* + \epsilon\sigma_1^* + \mathcal{O}(\epsilon^2). \tag{47}$$

We perform some preliminary computations of $A_{1,0}^* = A_1(a_0^*, \beta_0^*, \sigma_0^*),\ A_{2,0}^* = A_2(a_0^*, \beta_0^*, \sigma_0^*)$;

$$A_{1,0}^* = \frac{3d\lambda a_0^{*3}}{16\omega^2} + \frac{\sigma_0^* F_m \sin(\beta_0^*)}{4\omega^2} + \frac{9da_0^{*2} F_m \sin\beta_0^*}{32\omega^3}$$

$$A_{2,0}^* = -\frac{\lambda^2}{8\omega} - \frac{15d^2 a_0^{*4}}{256\omega^3} - \frac{5c^2 a_0^{*2}}{12\omega^3} - \frac{\lambda F_m \sin\beta_0^*}{8\omega^2 a_0}$$

then, we use the values of (46) and we get

$$\begin{aligned} A_{1,0}^* &= -\frac{F_m\sigma_0^*}{2\omega^2} = -\frac{\lambda a_0^*\sigma_0^*}{2\omega}, & \frac{\partial A_{1,0}^*}{\partial\sigma} &= \frac{F_m\sin(\beta_0^*)}{4\omega^2} = -\frac{a_0^*\lambda}{4\omega} \\ A_{2,0}^* &= -\frac{15d^2a_0^{*4}}{256\omega^3} - \frac{5c^2a_0^{*2}}{12\omega^3} = -\frac{5\sigma_0^{*2}}{12\omega} - \frac{5c^2a_0^{*2}}{12\omega^3}, & \frac{\partial A_{2,0}^*}{\partial\sigma} &= \frac{-F_m\cos(\beta_0^*)}{4\omega^2 a} = 0 \end{aligned} \tag{48}$$

$$\frac{\partial A_{1,0}^*}{\partial\beta} = \frac{\sigma F_m\cos(\beta_0^*)}{4\omega^2} - \frac{\lambda F_m \sin(\beta_0^*)}{8\omega^2} + \frac{9da^2F_m\cos(\beta_0^*)}{32\omega^3} = \frac{\lambda F_m}{8\omega^2} = \frac{\lambda^2 a_0^*}{8\omega} \tag{49}$$

$$\frac{\partial A_{2,0}^*}{\partial\beta} = -\frac{\sigma_0^*F_m\sin(\beta_0^*)}{4\omega^2 a_0^*} - \frac{3da_0^*F_m\sin(\beta_0^*)}{32\omega^3} - \frac{\lambda F_m\cos(\beta_0^*)}{8\omega^2 a} \tag{50}$$

$$= \frac{\sigma_0^*F_m}{4\omega^2a_0^*} + \frac{3da_0^*F_m}{32\omega^3} = \frac{\sigma_0^*F_m}{2\omega^2a_0^*} = \frac{\sigma_0^*\lambda}{2\omega}; \tag{51}$$

On the other hand, we notice that $\sin(\beta_0 + \epsilon\beta_1 + \mathcal{O}(\epsilon^2)) = -1 + \mathcal{O}(\epsilon^2)$ and with (46), we expand formula (41) to obtain at second order

$$\frac{\lambda a_1^*}{2} = A_{1,0}^* = -\frac{\lambda a_0^*\sigma_0^*}{2\omega}$$

and therefore

$$a_1^* = -\frac{a_0^*\sigma_0^*}{\omega} \tag{52}$$

We compute

$$\frac{\partial g_{1,0}^*}{\partial\sigma} = \epsilon\frac{\partial A_{1,0}^*}{\partial\sigma} + \mathcal{O}(\epsilon^2) = -\epsilon\frac{\lambda a_0}{4\omega} + O(\epsilon^2) \tag{53}$$

$$\frac{\partial g_{1,0}^*}{\partial\beta} = \frac{F_m\cos(\beta)}{2\omega} + \epsilon\frac{\partial A_{1,0}^*}{\partial\beta} + O(\epsilon^2) = -\epsilon\frac{F_m\beta_1^*}{2\omega} - \epsilon\frac{\lambda^2a_0^*}{8\omega} + \mathcal{O}(\epsilon^2); \tag{54}$$

where we have used $\cos(\beta_0 + \epsilon\beta_1 + \mathcal{O}(\epsilon^2)) = -\epsilon\beta_1 + \mathcal{O}(\epsilon^2)$ and $\frac{\partial a}{\partial\sigma} = 0$

$$\begin{aligned} \frac{dg_1}{d\sigma} &= \frac{\partial g_{1,0}^*}{\partial\sigma} + \frac{\partial g_{1,0}^*}{\partial\beta}\frac{\partial\beta}{\partial\sigma} + \frac{\partial g_{1,0}^*}{\partial\beta}\frac{\partial a}{\partial a} + \mathcal{O}(\epsilon^2) \\ &= \epsilon\left[-\frac{\lambda a_0^*}{4\omega} + \left(-\frac{F_m\beta_1^*}{2\omega} - \frac{\lambda^2a_0^*}{8\omega}\right)\left(-\frac{2}{\lambda}\right)\right] + \mathcal{O}(\epsilon^2) \\ &= -\epsilon a_0^*(\beta_1^* + \frac{\lambda}{2\omega}) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{55}$$

as $\frac{dg_1}{d\sigma} = 0$, we get

$$\beta_1^* = -\frac{\lambda}{2\omega}. \tag{56}$$

We use these approximations in the second equation (42) to obtain

$$-(\sigma_0^* + \epsilon\sigma_1^*) + \frac{3da_0^{*2}}{8\omega} + 6\epsilon\frac{da_0^* a_1^*}{8\omega} + \frac{F_m\beta_1^*}{2a_0^*\omega} + \epsilon A_{2,0}^* + \mathcal{O}(\epsilon^2) = 0 \tag{57}$$

and hence

$$\begin{aligned}
\sigma_1^* &= \frac{3da_0^* a_1^*}{4\omega} + \frac{F_m\beta_1^*}{2a_0^*\omega} + A_{2,0}^* && (58)\\
&= \frac{3da_0^*}{4\omega}\left(\frac{-a_0^*\sigma_0^*}{\omega}\right) + \frac{F_m}{2a_0^*\omega}\left(\frac{-\lambda}{2\omega}\right) + A_{2,0}^* \\
&= -2\frac{\sigma_0^2}{\omega} - \frac{\lambda^2}{4\omega} + A_{2,0}^* \\
&= -29\frac{\sigma_0^{*2}}{12\omega} - \frac{5c^2a_0^2}{12\omega^3} - \frac{\lambda^2}{4\omega}. && (59)
\end{aligned}$$

We remark that we get a frequency slightly different of the free vibration frequency associated to the same amplitude.

We have obtained the following important result.

Proposition 4 *The stationary solution of* (33) *satisfies*

$$\begin{cases} (\frac{F_m\sin(\beta)}{2\omega} - \frac{\lambda a}{2}) + \quad \epsilon A_1(a,\beta,\sigma) + \mathcal{O}(\epsilon^2) = 0 \\ (\sigma - \frac{3da^2}{8\omega} + \frac{F_m\cos(\beta)}{2a\omega}) + \quad \epsilon A_2(a,\beta,\sigma) + \mathcal{O}(\epsilon^2) = 0 \end{cases} \tag{60}$$

with

$$\begin{aligned}
A_1(a,\beta,\sigma) &= \frac{3d\lambda a^3}{16\omega^2} + \frac{\sigma F_m\sin\beta}{4\omega^2} + \frac{\lambda F_m\cos\beta}{8\omega^2} + \frac{9da_1^2F_m\sin\beta}{32\omega^3}\\
A_2(a,\beta,\sigma) &= -\frac{\lambda^2}{8\omega} - \frac{15d^2a^4}{256\omega^3} - \frac{5c^2a_1^2}{12\omega^3} + \frac{\sigma F_m\cos\beta}{4\omega^2 a_1} + c\frac{3daF_m\cos\beta}{32\omega^3} - \frac{\lambda F_m\sin\beta}{8\omega^2 a_1}
\end{aligned}$$

this stationary solution reaches its maximum amplitude for $\sigma = \sigma_0^* + \epsilon\sigma_1^* + \mathcal{O}(\epsilon^2)$, $a^* = a_0^* + \epsilon a_1^* + \mathcal{O}(\epsilon^2)$, $\beta^* = \beta_0^* + \epsilon\beta_1^* + \mathcal{O}(\epsilon^2)$ *with*

$$a_0^* = \frac{F_m}{\lambda\omega},\ \sigma_0^* = \frac{3da_0^{*2}}{8\omega} = \frac{3F_m^2}{8\lambda^2\omega^3},\quad \beta_0^* = -\frac{\pi}{2} \tag{61}$$

and

$$\sigma_1^* = -\frac{29}{12\omega}\sigma_0^{*2} - \frac{5c^2a_0^{*2}}{12\omega^3} - \frac{\lambda^2}{4\omega} = -\frac{87d^2a_0^4}{256\omega^3} - \frac{5c^2a_0^{*2}}{12\omega^3} - \frac{\lambda^2}{4\omega},\quad \beta_1^* = \frac{-\lambda}{2\omega},\quad a_1^* = -\frac{a_0^*\sigma_0^*}{\omega}$$

the periodic forcing is at the angular frequency

$$\tilde{\omega}_\epsilon = \omega + \epsilon\sigma_0^* + \epsilon^2\sigma_1^* + \mathcal{O}(\epsilon^2)$$

it is slightly different of the approximate angular frequency ν_ϵ of the undamped free periodic solution associated to the same amplitude. (14)*; for this frequency, the approximation (of the solution $\tilde{u} = \epsilon u$ of* (15) *up to the order ϵ^2) is periodic:*

$$\begin{cases} \tilde{u}(t) = \epsilon a^* \cos(\tilde{\omega}_\epsilon t + \beta^* t) \\ \qquad\qquad + \epsilon^2 [\frac{-ca^{*2}}{2\omega^2} + \frac{ca^{*2}}{6\omega^2} \cos(2(\tilde{\omega}_\epsilon t + \beta^*)) + \frac{da^{*3}}{32\omega^2} \cos(3(\tilde{\omega}_\epsilon t + \beta^*))] + \epsilon^3 r(\epsilon, t) \\ \tilde{u}(0) = \epsilon a^* + \epsilon^2 [\frac{-ca^{*2}}{3\omega^2} + \frac{da^{*3}}{32\omega^2}] + \mathcal{O}(\epsilon^3), \quad \dot{u}(0) = \mathcal{O}(\epsilon^3) \end{cases} \tag{62}$$

with r bounded in $C^2(0, t_\epsilon)$

Remark 11 We remark that, for ϵ small enough, this value of σ^* is indeed smaller than the maximal value that σ may reach in order that the previous expansion converges as indicated in proposition 3.

Remark 12 We have obtained an expansion of $\tilde{\omega}_\epsilon$ up to order ϵ^2 to be compared with the expansion with a double scale analysis (see in [BR13]); in particular the amplitude dependence on the frequency of the applied force depends on the ratio of c and d; see numerical results below.

We have justified the basic behaviour of a primary resonance; many other phenomena may appear like subharmonic resonances, see for example [Nay86].

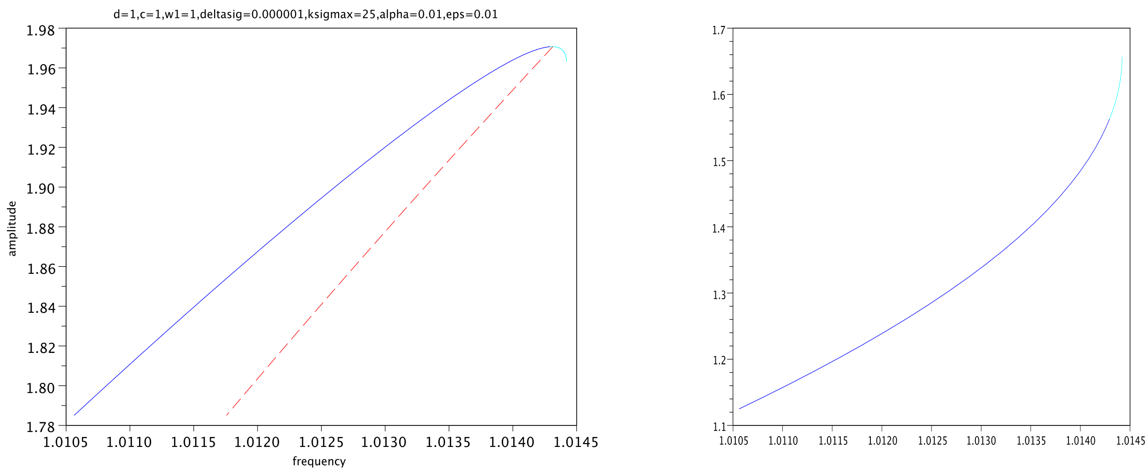

Fig. 2 Left: amplitude versus frequency of stationary forced solution in blue and magenta; amplitude of free solution in red. Right: phase versus frequency of stationary forced solution

In figure 2, we use $\epsilon = 0.01, \lambda = 1/2, c = 1, d = 1, \omega = 1, F = 1$. On the left, the solid line displays the amplitude of the solution of this equation with respect to values of the frequency; we have solved (40) with the routine FSOLVE of Scilab; it implements a variant of the hybrid method of Powell. In proposition 2, the solution is stable when sigma is small enough; the routine FSOLVE fails to solve the equation when σ is too large; then we have exchanged the use of σ and a. The dotted line plots the amplitude of the free solution with respect to its frequency. On the right, the phase $\gamma = -\beta$ is plotted with respect to the frequency; it is also obtained by solving (40) with the routine FSOLVE.

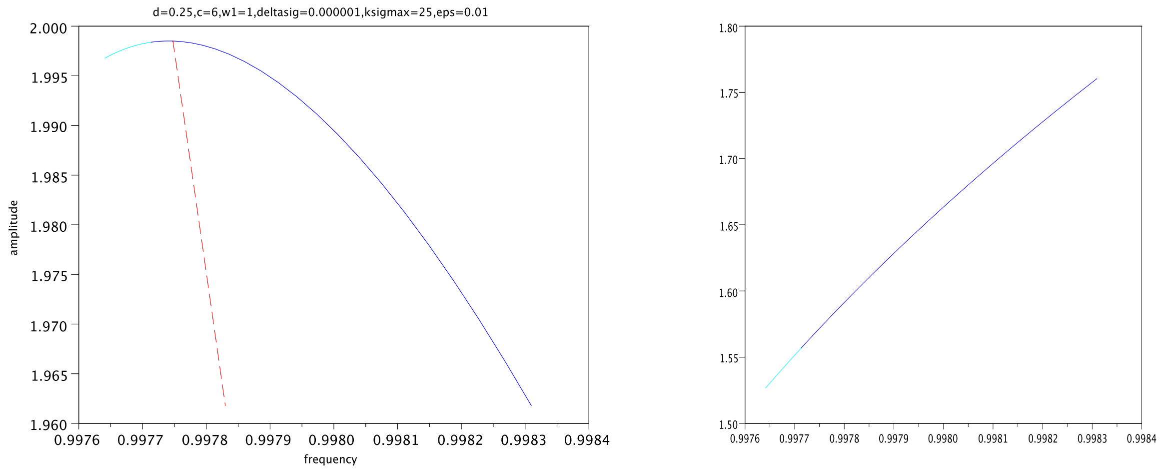

Fig. 3 Left: amplitude versus frequency of stationary forced solution in blue and magenta; amplitude of free solution in red. Right: phase versus frequency of stationary forced solution

In figure 3, we use $\epsilon = 0.01, \lambda = 1/2, c = 6, d = 1/4, \omega = 1, F = 1$. On the left the solid line displays the amplitude of the solution with respect to values of the frequency; on the right the phase γ is plotted. We notice that the behaviour is quite different of the previous plots.

Remark 13 We emphasise that the behaviour of the last plots is linked to the ration of c and d; this type of behaviour cannot be obtained with double scale expansion ; see [BR13].

3 System with local quadratic and cubic non linearity

3.1 Free vibrations, triple scale expansion up to second order

We consider a system of several vibrating masses attached to springs:

$$M\ddot{\tilde{u}} + K\,\tilde{u} + \Phi(\tilde{u}, \epsilon) = 0 \tag{63}$$

The mass matrix M and the rigidity matrix K are assumed to be symmetric and positive definite. We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the end points of some spring assumed to be non linear:

$$\Phi_{p-1}(\tilde{u}) = c(\tilde{u}_p - \tilde{u}_{p-1})^2 + \frac{d}{\epsilon}(\tilde{u}_p - \tilde{u}_{p-1})^3,\ \Phi_p = -\Phi_{p-1} \tag{64}$$

In order to get an approximate solution, we are going to display the equation in the generalised eigenvector basis:

$$K\phi_k = \omega_k^2 M\phi_k,\ \text{with}\ \phi_k^T\ M\ \phi_l = \delta\phi_{kl}, \quad k,l = 1\ldots,n \tag{65}$$

So we perform the change of functions:

$$\tilde{u} = \sum_{k=1}^{n} \tilde{y}_k\phi_k;\ K\tilde{u} = \sum_{k=1}^{n} \tilde{y}_k K\phi_k = \sum_{k=1}^{n} \tilde{y}_k\omega_k^2 M\phi_k;\ M\ddot{\tilde{u}} = \sum_{k=1}^{n} \ddot{\tilde{y}}_k M\phi_k \tag{66}$$

we obtain

$$\ddot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + \phi_k^T \Phi(\sum_{i=1}^{n} \tilde{y}_i \phi_i, \epsilon) = 0, \quad k = 1 \ldots, n$$

As Φ has only 2 components which are not zero, it can be written

$$\ddot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + (\phi_{k,p-1} - \phi_{k,p})\, \Phi_{p-1}(\sum_{i=1}^{n} \tilde{y}_i \phi_i, \epsilon) = 0, \quad k = 1 \ldots, n$$

or more precisely

$$\ddot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c \left(\sum_{i=1}^{n} \tilde{y}_i (\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \frac{d}{\epsilon} \left(\sum_{i=1}^{n} \tilde{y}_i (\phi_{i,p} - \phi_{i,p-1}) \right)^3 \right] = 0, \quad k = 1 \ldots, n \quad (67)$$

Remark 14 As we intend to look for a small solution, we consider a change of function $\boxed{\tilde{y}_k = \epsilon y_k}$ and we obtain the transformed equation:

$$\ddot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[\epsilon c \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \epsilon d \left(\sum_{i=1}^{n} y_i (\phi_{i,p} - \phi_{i,p-1}) \right)^3 \right] = 0, \quad k = 1 \ldots, n \quad (68)$$

3.1.1 Derivation of an asymptotic expansion

As for the 1 degree of freedom case, we use a triple scale expansion to compute an approximate small solution; more precisely, *we look for a solution close to a normal mode of the associated linear system*; we denote this mode by subscript ω_1; obviously by permuting the coordinates, this subscript could be anyone (different of p, this case would give similar results with slightly different formulae); we set

$$T_0 = \omega_1 t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t \text{ hence } \quad D_0 y_k = \frac{\partial y_k}{\partial T_0},\ D_1 y_k = \frac{\partial y_k}{\partial T_1} \text{ and } D_2 y_k = \frac{\partial y_k}{\partial T_2} \quad (69)$$

and we use the *ansatz*:

$$y_k(t) = y_k(T_0, T_1, T_2) = y_k^{(1)}(T_0, T_1, T_2) + \epsilon y_k^{(2)}(T_0, T_1, T_2) + \epsilon^2 r_k(T_0, T_1, T_2) \quad (70)$$

So we have:

$$\begin{aligned} \frac{d^2 y_k}{dt^2} &= \omega_1^2 D_0^2 y_k^{(1)} + \epsilon \left[2\omega_1 D_0 D_1 y_k^{(1)} + D_0^2 y_k^{(2)} \right] \\ &+ \epsilon^2 \left[2\omega_1 D_0 D_2 y_k^{(1)} + D_1^2 y_k^{(1)} + 2\omega_1 D_0 D_1 y_k^{(2)} + D_0^2 r \right] \\ &+ \epsilon^3 \left[2 D_1 D_2 y_k^{(1)} + 2\omega_1 D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} + \mathcal{D}_3 r_k \right] \\ &+ \epsilon^4 \left[D_2^2 y_k^{(1)} + 2 D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)} \right] \end{aligned} \quad (71)$$

with

$$\mathcal{D}_3 r_k = \frac{1}{\epsilon}\left(\frac{d^2 r_k}{dt^2} - \omega_1^2 D_0^2 r_k\right) = 2\omega_1 D_0 D_1 r_k + \epsilon[2\omega_1 D_0 D_2 r_k + D_1^2 r_k] + 2\epsilon^2 D_1 D_2 r_k + \epsilon^3 D_2^2 r_k$$

We plug previous expansions (70) and (71) into (68); by identifying the coefficients of the powers of ϵ in the expansion of (68), we get:

$$\begin{cases} \omega_1^2 D_0^2 y_k^{(1)} + \omega_k^2 y_k^{(1)} = 0 \quad , & k = 1\ldots, n \\ \omega_1^2 D_0^2 y_k^{(2)} + \omega_k^2 y_k^{(2)} = S_{2,k} \quad , & k = 1\ldots, n \\ \omega_1^2 D_0^2 r_k + \omega_k^2 r_k = S_{3,k} \quad , & k = 1\ldots, n \end{cases} \tag{72}$$

where $S_{2,k}, S_{3,k}$ are defined below; to simplify the manipulations, we set $\delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1})$;

$$S_{2,k} = -c\delta\phi_{kp}\left(\sum_{l,m} y_l^{(1)}\delta\phi_{lp} y_m^{(1)}\delta\phi_{mp}\right) - d\delta\phi_{kp}\left(\sum_{g,l,o} y_g^{(1)} y_l^{(1)}\delta\phi_{lp} y_o^{(1)}\delta\phi_{gp}\delta\phi_{op}\right) - 2\omega_1 D_0 D_1 y_k^{(1)}$$

$$\begin{aligned} S_{3,k} = -c\delta\phi_{kp}\left(\sum_{l,j} y_l^{(1)} y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) - d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(1)} y_l^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\ - 2\omega_1 D_0 D_2 y_k^{(1)} - D_1^2 y_k^{(1)} - 2\omega_1 D_0 D_1 y_k^{(2)} - \epsilon R_k(y_1^{(1)}, y_1^{(2)}, r_k, \epsilon) \end{aligned}$$

with

$$\begin{aligned} R_k(\epsilon, r_k, y_k^{(1)}, y_k^{(2)}) &= 2D_1 D_2 y_k^{(1)} + 2\omega_1 D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} \\ &+ c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(2)} y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) + c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(1)} r_l\delta\phi_{jp}\delta\phi_{lp}\right) \\ &+ d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(2)} y_l^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) + d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(1)} r_l\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\ &+ \mathcal{D}_3 r_k + \epsilon(D_2^2 y_k^{(1)} + 2D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)}) + \epsilon\rho(y_k^{(1)}, y_k^{(2)}, r_k, \epsilon) \end{aligned}$$

and with a polynomial in the variables r_n with coefficients $y_l^{(1)}, y_m^{(2)}$,

$$\begin{aligned}
\rho(y_k^{(1)}, y_k^{(2)}, r_k, \epsilon) = c\delta\phi_{kp}\left(\sum_{l,j} y_l^{(2)} r_j \delta\phi_{lp}\delta\phi_{jp}\right) + d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} y_g^{(2)} r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\
+ d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(2)} y_g^{(2)} y_l^{(2)} \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\
+ \epsilon c\Bigg[c\delta\phi_{kp}\left(\sum_{l,j} r_l r_j \delta\phi_{lp}\delta\phi_{jp}\right) + d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(2)} y_g^{(2)} r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \\
+ d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(1)} r_g r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right)\Bigg] \\
+ \epsilon^2 d\delta\phi_{kp}\left(\sum_{h,g,l} y_h^{(2)} r_g r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) + \epsilon^3 d\delta\phi_{kp}\left(\sum_{h,g,l} r_h r_g r_l \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{lp}\right) \quad (73)
\end{aligned}$$

We set $\theta(T_0, T_1, T_2) = T_0 + \beta_1(T_1, T_2)$; we note that $D_0\theta = 1,\ D_1\theta = D_1\beta$ and $D_2\theta = D_2\beta_1$; we solve the first set of equations (72), imposing $O(\epsilon^3)$ initial Cauchy data for $k \neq 1$ and $D_0 y_1^{(1)}(0) = 0$; we get:

$$\begin{cases} y_1^{(1)} = a_1(T_1, T_2)\ \cos(\theta) \\ y_k^{(1)} = 0, \qquad k = 2 \ldots n \end{cases} \quad (74)$$

Remark 15 We note that a_1 and β_1 are not constants but functions of times T_1 and T_2 because u depends on these times scales. The dependence of these functions with respect to T_1 and T_2 will be determined by solving the equations of the following orders and eliminating secular terms.

First, we determine the dependence in T_1; we manipulate the right hand sides:

$$\begin{aligned}
S_{2,1} = -\delta\phi_{1p}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big] \\
+ 2\omega_1\left[a_1 D_1\beta_1 \cos(\theta) + D_1 a_1 \sin(\theta)\right]
\end{aligned}$$

$$S_{2,k} = -\delta\phi_{kp}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big],\ \text{for } k \neq 1$$

In $S_{2,1}$, we gather the terms at angular frequency ω_1;

$$S_{2,1} = -3\frac{da_1^3}{4}\cos(\theta))\delta\phi_{1p}^4 + 2\omega_1\left[a_1 D_1\beta_1 \cos(\theta) + D_1 a_1 \sin(\theta)\right] + S_2^\sharp \quad (75)$$

with

$$S_{2,1}^\sharp = -\delta\phi_{1p}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}\cos(3\theta)\delta\phi_{1p}^3\Big]$$

It appears some terms at the frequency of the system, these terms provide a solution $y_1^{(2)}$ of the equation (72) which is non periodic and non bounded over long time intervals. We will eliminate these terms by imposing:

$$\begin{cases} D_1 a_1 = 0 \\ D_1\beta_1 = \frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} \end{cases} \tag{76}$$

and if we assume that ω_1^2 is a simple eigenvalue and $\omega_k^2 \neq 9\omega_1^2$, $\omega_k^2 \neq 4\omega_1^2$ (no internal resonance), the solution of the second equation (72) is:

$$\begin{cases} y_1^{(2)} = \delta\phi_{1p}^3[-\frac{ca_1^2}{2\omega_1^2} + \frac{ca_1^2}{6\omega_1^2}\cos(2\theta)] + \delta\phi_{1p}^4\frac{da_1^3}{32\omega_1^2}\cos(3\theta) \\ y_k^{(2)} = \delta\phi_{kp}\delta\phi_{1p}^2[-\frac{ca_1^2}{2\omega_k^2} + \frac{ca_1^2}{2(4\omega_1^2-\omega_k^2)}\cos(2\theta)] + \delta\phi_{kp}\delta\phi_{1p}^3\frac{da_1^3}{4(9\omega_1^2-\omega_k^2)}\cos(3\theta), \quad k = 2,\dots,n. \end{cases} \tag{77}$$

where we have omitted the term at angular frequency ω_1 which is redundant with $y_1^{(1)}$. For the third set of equations of (72), r is the unknown, this equation contains non-linearities, we do not solve it but we show that the solution is bounded on an interval dependent of ϵ. The right hand side, after some manipulations is:

$$\begin{aligned} S_{3,1} = \sin(\theta)\big(2\omega_1 D_2 a_1 + 2D_1 a_1 D_1\beta_1 + a_1 D_1^2\beta_1\big) \\ \cos(\theta)\left(2\omega_1 a_1 D_2\beta_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}\right) \\ + S_{3,1}^\sharp - \epsilon R_1(r_1,\epsilon,y_1^{(1)},y_1^{(2)}) \end{aligned}$$

where

$$\begin{aligned} S_{3,1}^\sharp = \frac{5cd\delta\phi_{1p}^7 a_1^4}{8\omega_1^2} + \sin 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1}{3\omega_1}D_1 a_1\right] + \cos 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1^2}{3\omega_1}D_1\beta_1 + \frac{15cd\delta\phi_{1p}^7 a_1^4}{32\omega_1^2}\right] \\ + \sin 3\theta\left[\frac{9d\delta\phi_{1p}^4 a_1^2}{16\omega_1}D_1 a_1\right] + \cos 3\theta\left[-\frac{c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{64\omega_1^2} + \frac{9da_1^3\delta\phi_{1p}^4}{16\omega}D_1\beta_1\right] \\ + \cos 4\theta\left[-\frac{5cd\delta\phi_{1p}^7 a_1^4}{32\omega_1^2}\right] - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}\cos 5\theta \end{aligned} \tag{78}$$

and

$$S_{3,k} = \cos(\theta)\left(\frac{5c^2\delta\phi_{kp}\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{kp}\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}\right) + S_{3,k}^\sharp - \epsilon R_k(r_k,\epsilon,y_1^{(1)},y_1^{(2)})$$

where

$$\begin{aligned} S_{3,k}^\sharp = \frac{5cd\delta\phi_{kp}\delta\phi_{1p}^6 a_1^4}{8\omega_1^2} + \sin 2\theta\left[\frac{4c\delta\phi_{kp}\delta\phi_{1p}^2 a_1}{3\omega_1}D_1 a_1\right] + \cos 2\theta\left[\frac{4c\delta\phi_{kp}\delta\phi_{1p}^2 a_1^2}{3\omega_1}D_1\beta_1 + \frac{15cd\delta\phi_{kp}\delta\phi_{1p}^6 a_1^4}{32\omega_1^2}\right] \\ + \sin 3\theta\left[\frac{9d\delta\phi_{kp}\delta\phi_{1p}^3 a_1^2}{16\omega_1}D_1 a_1\right] + \cos 3\theta\left[-\frac{c^2\delta\phi_{kp}\delta\phi_{1p}^5 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{kp}\delta\phi_{1p}^7 a_1^5}{64\omega_1^2} + \frac{9d\delta\phi_{kp}\delta\phi_{11}^3 a_1^3}{16\omega_1}D_1\beta_1\right] \\ + \cos 4\theta\left[-\frac{5cd\delta\phi_{kp}\delta\phi_{1p}^6 a_1^4}{32\omega_1^2}\right] - \frac{3d^2\delta\phi_{kp}\delta\phi_{1p}^7 a_1^5}{128\omega_1^2}\cos 5\theta \end{aligned}$$

By imposing

$$\left\{\begin{array}{l} 2\omega_1 D_2 a_1 + 2D_1 a_1 D_1 \beta_1 + a_1 D_1^2 \beta_1 = 0 \\ 2\omega_1 a_1 D_2 \beta_1 - D_1^2 a_1 + a_1 (D_1\beta_1)^2 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2} = 0 \end{array}\right.$$

we get that $S_{3,1} = S^\sharp_{3,1} - \epsilon R_1(r_1, \epsilon, y_1^{(1)}, y_1^{(2)})$ contains no terms at the frequency of the system.

As $D_1 a_1 = 0$ and $D_1\beta_1 = \frac{-3d\delta\phi_{1p}^4 a_1^2}{8\omega_1}$, we obtain

$$2\omega_1 a_1 D_2\beta_1 + a_1\Big(\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1}\Big)^2 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2} = 0$$

so:

$$D_2 a_1(T_2) = 0 \qquad \text{and} \qquad D_2\beta_1(T_2) = -\frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} \tag{79}$$

As a, β do not depend on T_0,

$$\left\{\begin{array}{l} \frac{da_1}{dt} = \epsilon D_1 a_1 + \epsilon^2 D_2 a_1 + \mathcal{O}(\epsilon^3) \\ \frac{d\beta}{dt} = \epsilon D_1\beta + \epsilon^2 D_2\beta + \mathcal{O}(\epsilon^3) \end{array}\right. \tag{80}$$

and taking into account (76) and (79),we obtain:

$$\frac{da_1}{dt} = 0 \quad \text{and} \quad \frac{d\beta_1}{dt} = \epsilon\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} + \epsilon^2\Big(\frac{-5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3}\Big) \tag{81}$$

As a result, the solution of these equations is:

$$a_1 = cte \qquad \text{and} \qquad \beta_1 = \left[\epsilon\frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} + \epsilon^2\Big(-\frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3}\Big)\right] t \tag{82}$$

In order to show that r_1 is bounded, after eliminating the secular terms, we can go back to the variable t in the equation of r_k, we get:

$$\frac{d^2 r_1}{dt^2} + \omega_1^2 r_1 = \tilde{S}_{3,1} \quad \text{with} \quad \tilde{S}_{3,1} = S^\sharp_{3,1}(t,\epsilon) - \epsilon\tilde{R}_1(r_1, \epsilon, y_1^{(1)}, y_1^{(2)})$$
$$\frac{d^2 r_k}{dt^2} + \omega_1^2 r_k = \tilde{S}_{3,k} \quad \text{with} \quad \tilde{S}_{3,k} = S^\sharp_{3,k}(t,\epsilon) - \epsilon\tilde{R}_k(r_k, \epsilon, y_k^{(1)}, y_k^{(2)}) \quad k = 2, \dots, n$$

where $S^\sharp_{3,1}$ is in (78) where all time scales T_0, T_1, T_2 are expressed with the time variable t.

$$\tilde{R}_1 = R_1(\epsilon, r_1, y_1^{(1)}, y_1^{(2)}) - \mathcal{D}_3 r_1$$

After these manipulations, we can state a proposition which will be easily proved with technical lemmas of the Appendix.

Proposition 5 *We assume that ω_1^2 is a simple eigenvalue and $\omega_k^2 - 9\omega_1^2 \neq 0$, $\omega_k^2 - 4\omega_1^2 \neq 0$ (no internal resonance), then it exists $\varsigma > 0$ such that for all $t \leq t_\epsilon = \frac{\varsigma}{\epsilon^2}$, the solution $\tilde{y}_k = \epsilon y_k$ of* (67) *with the initial data*

$$\begin{array}{l} \tilde{y}_1(0) = \epsilon a_1 + \epsilon^2\ (-\frac{\check{c}_1 a_1^2}{3\omega_1^2} + \frac{\check{d}_1 a_1^3}{32\omega_1^2}) + \epsilon^3 r_1(\epsilon, 0), \quad \dot{\tilde{y}}_1(0) = \mathcal{O}(\epsilon) \\ \tilde{y}_k(0) = \epsilon^2[-\frac{\check{c}_k a_1^2}{2\omega_k^2} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2 - \omega_k^2)} + \frac{\check{d}_k a_1^3}{4(9\omega_1^2 - \omega_k^2)}] + \epsilon^3 r_k(\epsilon, 0), \quad \dot{\tilde{y}}_k(0) = \mathcal{O}(\epsilon) \end{array} \tag{83}$$

has the following expansion:

$$\begin{cases} \tilde{y}_1(t) = \epsilon a_1 \cos(\nu_\epsilon t) + \epsilon^2[-\frac{\check{c}_1 a_1^2}{2\omega_1^2} + \frac{\check{c}_1 a_1^2}{6\omega_1^2}\cos(2(\nu_\epsilon t)) + \frac{\check{d}_1 a_1^3}{32\omega_1^2}\cos(3(\nu_\epsilon t))] + \epsilon^3 r_1(\epsilon, t) \\ \tilde{y}_k(t) = \epsilon^2[-\frac{\check{c}_k a_1^2}{2\omega_k^2} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2-\omega_k^2)}\cos(2(\nu_\epsilon t)) + \frac{\check{d}_k a_1^3}{4(9\omega_1^2-\omega_k^2)}\cos(3(\nu_\epsilon t))] + \epsilon^3 r_k(\epsilon, t) \end{cases} \tag{84}$$

with r_k uniformly bounded in $C^2(0, t_{\epsilon^2})$ for $k = 1, \dots n$ and the angular frequency

$$\nu_\epsilon = \omega_1 + \epsilon(\frac{3\check{d}_1 a_1^2}{8\omega_1}) + \epsilon^2\left(\frac{-5\check{c}_1^2 a_1^2}{12\omega_1^3} - \frac{15\check{d}_1^2 a_1^4}{256\omega_1^3}\right) + \mathcal{O}(\epsilon^3) \tag{85}$$

with $\delta\phi_{1p} = (\phi_{1,p} - \phi_{1,p-1}), \delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1})$, $\check{c}_1 = c(\delta\phi_{1p})^3$, $\check{d}_1 = d(\delta\phi_{1p})^4$ and

$$\check{c}_k = c(\delta\phi_{1p})^2\delta\phi_{kp},\ \check{d}_k = d(\delta\phi_{1p})^3\delta\phi_{kp}$$

Corollary 1 *The solution of (63) with initial conditions*

$$\begin{aligned} {}^t\phi_1\tilde{u}(0) &= \epsilon a_1 + \epsilon^2\ (-\frac{\check{c}a_1^2}{3\omega_1^2} + \frac{\check{d}a_1^3}{32\omega_1^2}) + \epsilon^3 r_1(\epsilon, 0), \quad {}^t\phi_1\dot{u}(0) = \mathcal{O}(\epsilon^2) \\ {}^t\phi_k\tilde{u}(0) &= \epsilon^2[-\frac{\check{c}_k a_1^2}{2\omega_k^2} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2-\omega_k^2)} + \frac{\check{d}_k a_1^3}{4(9\omega_1^2-\omega_k^2)}] + \epsilon^3 r_k(\epsilon, 0), \quad {}^t\phi_k\dot{u}(0) = \mathcal{O}(\epsilon^2) \end{aligned} \tag{86}$$

$$\text{is } \tilde{u}(t) = \sum_{k=1}^{n} \tilde{y}_k(t)\phi_k + \epsilon^3 r(t, \epsilon) \tag{87}$$

with the expansion of y_k of previous proposition.

Proof For the proposition, we use lemma 4; set $S_1 = \tilde{S}_{3,1}$, $S_k = S_{3,k}$ for $k = 1, \dots n$; as we have enforced (82), the functions S_k are periodic, bounded, and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$; then S satisfies satisfies the lemma hypothesis. Similarly, set $g = \tilde{R}$, its components are polynomials in r with coefficients which are bounded functions, so it is lipschitzian on the bounded subsets it satisfies the hypothesis of the lemma and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (66) □

Remark 16 1. We have obtained a periodic asymptotic expansion of a solution of system (63); they are called non linear normal modes in the mechanical community ([KPGV09, **?**]. If the initial condition is close to an eigenvector Φ_1 up to second order, the component of the solution on this eigenvector has an approximation which has the same form as for the single degree of freedom system; the other components remain small.
2. The frequency shift is given by a similar formula with c replaced by $\check{c} = c(\phi_{1,p} - \phi_{1,p-1})^3$, d replaced by $\check{d} = d(\phi_{1,p} - \phi_{1,p-1})^4$; so the frequency shift depends on the position of non-linearity with respect to the components of the associated eigenvector.
3. In the spirit of inverse problems, this previous point opens a way to localise the non-linearity.
4. *We do not study the periodicity of the solution itself but as the system is Hamiltonian, it could be obtained from general results, for example see [MH92].*
5. In the next section, under the assumption of no internal resonance, we shall derive that the frequencies of the normal mode are close to resonant frequencies for an associated forced system, the so called primary resonance; with some changes, secondary resonance could be derived along similar lines.

3.1.2 Numerical results

We consider numerical solution of (63) with (64); we have chosen $M = I$; $u = 0$ at both ends, so K is the classical matrix

$$k \begin{pmatrix} 2 & -1 \dots\dots\dots \\ -1 & 2 & -1 \dots\dots \\ 0 & -1 & 2 & -1 \dots \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots & -1 & 2 \end{pmatrix};$$

$C = \lambda I$ with $\lambda = 1/2$; for numerical balance, we have computed $\frac{u}{\epsilon}$; with the choice $p = 1$ we have $\Phi_1 = \epsilon[cu_1^2 + du_1^3]$ with $c = 1, d = 1$. In figure 3.1.2, for 29 degrees of freedom, we find the Fourier transform of the components; some components have the same transform; the graphs are slightly non symmetric; we find also several curves in phase space for some components of the system.

We remark that up to numerical integration errors, all frequencies are equal and the components are periodic. All these characteristics are coherent with the results obtained by asymptotic expansions: an approximation of a non linear normal mode which is a continuation with respect to ϵ of a linear normal mode.

3.2 Forced, damped vibrations, triple scale expansion

3.2.1 Derivation of an asymptotic expansion

We consider a similar system of forced vibrating masses attached to springs with some damping and submitted to a periodic forcing:

$$M\ddot{\tilde{u}} + \epsilon C\dot{\tilde{u}} + Ku + \Phi(\tilde{u}, \epsilon) = \epsilon^2 F \cos\tilde{\omega_\epsilon} t \tag{88}$$

with the same assumptions as in subsection 3.1. We assume that the non linearity is local, all components are zero except for two components $p-1,\ p$ which correspond to the endpoints of some spring assumed to be non linear. As for free vibrations, we perform the change of function

$$\tilde{u} = \sum_{k=1}^{n} \tilde{y}_k \phi_k \tag{89}$$

with ϕ_k, the generalised eigenvectors of (65). However, the distribution of damping is almost always unknown and it is usually necessary to make an assumption about its distribution; a simple and widely used hypothesis is to choose a modal damping (hypothesis of Basile in french terminology):

$$C = \epsilon_M M + \epsilon_K K$$

Therefore

$$\ddot{\tilde{y}}_k + \epsilon\lambda_k\dot{\tilde{y}}_k + \omega_k^2\tilde{y}_k + {}^t\phi_k\Phi(\sum_{i=1}^{n}\tilde{y}_i\phi_i, \epsilon) = \epsilon^2 f_k \cos\tilde{\omega_\epsilon} T_0, \qquad k = 1\dots, n$$

with

$$\epsilon_M + \epsilon_K\omega_k^2 = \lambda_k \quad \textit{and} \quad {}^t\phi_k F = f_k$$

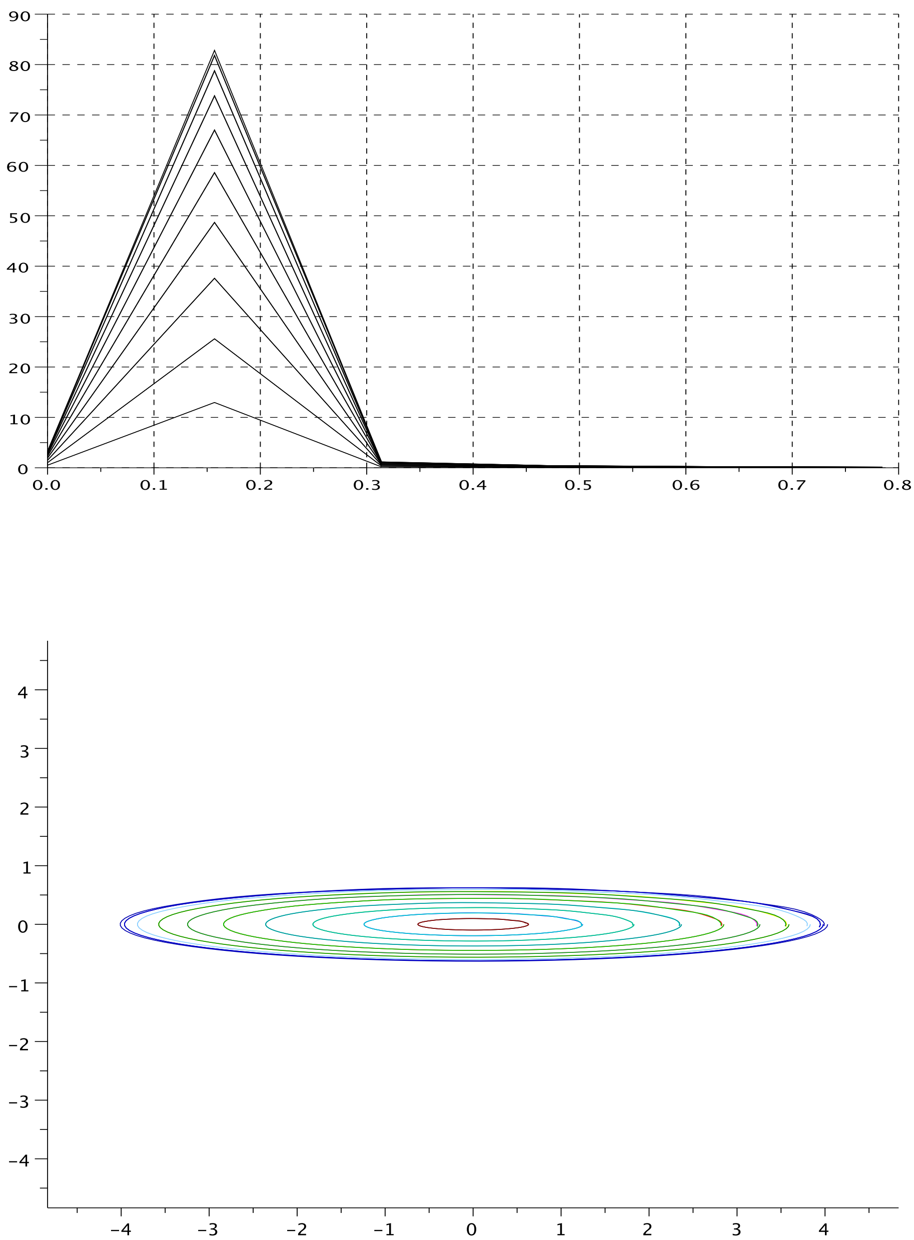

Fig. 4 Absolute value of the Fourier transform for (fft) (left); phase portrait(right)

As for the free vibration case, Φ has only 2 components which are not zero, so the system can be written:

$$\ddot{\tilde{y}}_k + \epsilon\lambda_k \dot{\tilde{y}}_k + \omega_k^2 \tilde{y}_k + (\phi_{k,p-1} - \phi_{k,p}) \left[c \left(\sum_{i=1}^{n} \tilde{y}_i(\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \frac{d}{\epsilon} \left(\sum_{i=1}^{n} \tilde{y}_i(\phi_i - \phi_{i,p-1}) \right)^3 \right]$$
$$= \epsilon^2 f_k \cos \tilde{\omega_\epsilon} T_0, \quad \mathit{for} \quad k = 1 \ldots, n \quad (90)$$

Remark 17 As we intend to look for a small solution, we consider a change of function $\boxed{\tilde{y}_k = \epsilon y_k}$ and we obtain the transformed equation:

$$\ddot{y}_k + \epsilon\lambda_k \dot{y}_k + \omega_k^2 y_k + (\phi_{k,p-1} - \phi_{k,p}) \left[\epsilon c \left(\sum_{i=1}^{n} y_i(\phi_{i,p} - \phi_{i,p-1}) \right)^2 + \epsilon d \left(\sum_{i=1}^{n} y_i(\phi_i - \phi_{i,p-1}) \right)^3 \right]$$
$$= \epsilon f_k \cos(\tilde{\omega}_\epsilon t) \quad \mathit{for} \quad k = 1 \ldots, n \quad (91)$$

We will highlight a link between the frequency of the free solution of the preceding paragraph and the amplitude of the steady state forced solution; it is assumed that the excitation frequency is close to the natural frequency of the linear system

$$\tilde{\omega_\epsilon} = \omega_1 + \epsilon\sigma \quad (92)$$

As in the previous case, we look for a small solution with a triple scale expansion, more precisely, we look for **a periodic solution close to an eigenmode of the linear system**, for example, we consider mode y_1 (by permuting the indexes it could be any mode); we set:

$$T_0 = \tilde{\omega}_\epsilon t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t \text{ hence } \quad D_0 y_k = \frac{\partial y_k}{\partial T_0}, \quad D_1 y_k = \frac{\partial y_k}{\partial T_1} \text{ and } D_2 y_k = \frac{\partial y_k}{\partial T_2}$$

Derivatives of y_k may be expanded:

$$\frac{dy_k}{dt} = \tilde{\omega}_\epsilon D_0 y_k + \epsilon D_1 y_k + \epsilon^2 D_2 y_k \quad (93)$$

and

$$\frac{d^2 y_k}{dt^2} = \tilde{\omega}_\epsilon^2 D_0^2 y_k + 2\epsilon\tilde{\omega}_\epsilon D_0 D_1 y_k + 2\epsilon^2 D_0 D_2 y_k + \epsilon^2 D_1^2 y_k + 2\epsilon^3 D_1 D_2 y_k + \epsilon^4 D_2^2 y_k \quad (94)$$

we use the *ansatz*

$$y_k(t) = y_k(T_0, T_1, T_2) = y_k^{(1)}(T_0, T_1, T_2) + \epsilon y_k^{(2)}(T_0, T_1, T_2) + \epsilon^2 r_k(T_0, T_1, T_2) \quad (95)$$

we get:

$$\begin{aligned}\frac{dy_k}{dt} &= \frac{dy_k^{(1)}}{dt} + \epsilon \frac{dy_k^{(2)}}{dt} + \epsilon^2 \frac{dr_k}{dt} = \epsilon \frac{dy_k^{(1)}}{dt} + \epsilon^2 \frac{dy_k^{(2)}}{dt} + \epsilon^2 D_0 r_k + \epsilon^2 (\frac{dr_k}{dt} - D_0 r_k) \\ &= [\tilde{\omega}_\epsilon D_0 y_k^{(1)} + \epsilon D_1 y_k^{(1)} + \epsilon^2 D_2 y_k^{(1)}] + \epsilon[\tilde{\omega}_\epsilon D_0 y_k^{(2)} + \epsilon D_1 y_k^{(2)} + \epsilon^2 D_2 y_k^{(2)}] \\ &\qquad + \epsilon^2 \tilde{\omega}_\epsilon D_0 r_k + \epsilon^2 (\frac{dr_k}{dt} - \tilde{\omega}_\epsilon D_0 r_k)\end{aligned}$$

we note that $\frac{dr_k}{dt} - \tilde{\omega}_\epsilon D_0 r_k = \epsilon D_1 r_k + \epsilon^2 D_2 r_k$; it is of order 1 in ϵ. For the second derivative, as in the case of free vibration, we introduce:

$$\begin{aligned}\mathcal{D}_3 r_k &= \frac{1}{\epsilon}\left(\frac{d^2 r_k}{dt^2} - \tilde{\omega}_\epsilon^2 D_0^2 r_k\right)\\ &= 2\tilde{\omega}_\epsilon D_0 D_1 r_k + \epsilon\left[2\tilde{\omega}_\epsilon D_0 D_2 r_k + D_1^2 r_k + 2 D_2 D_1 r_k\right] + \epsilon^3 D_2^2 r_k\end{aligned}$$

$$\begin{aligned}\frac{d^2 y_k}{dt^2} &= \frac{d^2 y_k^{(1)}}{dt^2} + \epsilon\frac{d^2 y_k^{(2)}}{dt^2} + \epsilon^2\frac{d^2 r_k}{dt^2} = \frac{d^2 y_k^{(1)}}{dt^2} + \epsilon^1\frac{d^2 y_k^{(2)}}{dt^2} + \epsilon^2\tilde{\omega}_\epsilon D_0^2 r_k + \epsilon^3 \mathcal{D}_3 r_k\\ &= \tilde{\omega}_\epsilon^2 D_0^2 y_k^{(1)} + \epsilon\left[2\tilde{\omega}_\epsilon D_0 D_1 y_k^{(1)} + D_0^2 y_k^{(2)}\right]\\ &\quad + \epsilon^2\left[2\tilde{\omega}_\epsilon D_0 D_2 y_k^{(1)} + D_1^2 y_k^{(1)} + 2\tilde{\omega}_\epsilon D_0 D_1 y_k^{(2)} + D_0^2 r_k\right]\\ &\quad + \epsilon^3\left[2 D_1 D_2 y_k^{(1)} + 2\tilde{\omega}_\epsilon D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} + \mathcal{D}_3 r_k\right]\\ &\quad + \epsilon^4\left[D_2^2 y_k^{(1)} + 2 D_1 D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)}\right]\end{aligned}$$

We plug previous expansions (93), (95) and (94) of y^k into (91); by identifying the coefficients of the powers of ϵ, we get:

$$\left\{\begin{array}{lll}\omega_1^2 D_0^2 y_k^{(1)} + \omega_k^2 y_k^{(1)} = 0 & , & k = 1\dots, n\\ \omega_1^2 D_0^2 y_k^{(2)} + \omega_k^2 y_k^{(2)} = S_{2,k} & , & k = 1\dots, n\\ \omega_1^2 D_0^2 r + \omega_k^2 r = S_{3,k} & , & k = 1\dots, n\end{array}\right. \tag{96}$$

with

$$\begin{aligned}S_{2,k} = &-c\delta\phi_{kp}\left(\sum_{l,m} y_l^{(1)}\delta\phi_{lp} y_m^{(1)}\delta\phi_{mp}\right) - d\delta\phi_{kp}\left(\sum_{g,n,o} y_g^{(1)}\delta\phi_{gp} y_n^{(1)}\delta\phi_{np} y_o^{(1)}\delta\phi_{op}\right)\\ &- 2\omega_1 D_0 D_1 y_k^{(1)} - \lambda_k\omega_1 D_0 y_k^{(1)} - 2\omega\sigma D_0^2 y_k^{(1)} + f_k\cos(T_0),\end{aligned}$$

$$\begin{aligned}S_{3,k} = &-c\delta\phi_{kp}\left(\sum_{l,j} y_l^{(1)} y_j^{(2)}\delta\phi_{lp}\delta\phi_{jp}\right) - d\delta\phi_{kp}\left(\sum_{h,g,n} y_h^{(1)} y_g^{(1)} y_n^{(2)}\delta\phi_{hp}\delta\phi_{gp}\delta\phi_{np}\right)\\ &- 2\omega_1 D_0 D_2 y_k^{(1)} - D_1^2 y_k^{(1)} - 2\omega_1 D_0 D_1 y_k^{(2)} - \sigma^2 D_0^2 y_k^{(1)} - 2\omega_1\sigma D_0^2 y_k^{(1)} - 2\sigma D_0 D_1 y_k^{(1)} - 2\omega_1\sigma D_0^2 y_k^{(2)}\\ &- \lambda_k D_1 y_k^{(1)} - -\lambda_k\sigma D_0 y_k^{(1)} - \lambda_k\omega_1 D_0 y_k^{(2)} - \epsilon R_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)})\end{aligned}$$

where $\delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1})$ and with

$$
\begin{aligned}
R_k(\epsilon, r_k, y_k^{(1)}, y_k^{(2)}) &= 2D_1D_2y_k^{(1)} + 2\omega_1 D_0 D_2 y_k^{(2)} + D_1^2 y_k^{(2)} \\
&+ c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(2)} y_j^{(2)} \delta\phi_{lp}\delta\phi_{jp}\right) + c\delta\phi_{kp}\left(\sum_{l,j} y_j^{(1)} r_l \delta\phi_{jp}\delta\phi_{lp}\right) \\
&+ d\delta\phi_{kp}\left(\sum_{h,g,n} y_h^{(1)} y_g^{(2)} y_n^{(2)} \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{np}\right) + d\delta\phi_{kp}\left(\sum_{h,g,n} y_h^{(1)} y_g^{(1)} r_n \delta\phi_{hp}\delta\phi_{gp}\delta\phi_{np}\right) \\
&\lambda_k(\omega_1 D_0 r + D_2 y_k^{(1)} + D_1 y_k^{(2)} + \epsilon D_2 y_k^{(2)}) + \mathcal{D}_3 r \\
&+ \epsilon\left(D_2^2 y_k^{(1)} + 2D_1D_2 y_k^{(2)} + \epsilon D_2^2 y_k^{(2)}\right) + \lambda_k(\frac{dr}{dt} - \omega_1 D_0 r) + \epsilon\rho(y_k^{(1)}, y_k^{(2)}, r_k, \epsilon)
\end{aligned}
$$

and the polynomial ρ displayed in (73).

We solve the first set of equations (96) imposing initial Cauchy data for $k \neq 1$ of order $\mathcal{O}(\epsilon^2)$ and $D_0 y_1^{(1)}(0) = 0$ we get:

$$
\begin{cases} y_1^{(1)} = a_1(T_1, T_2)\ \cos(\theta) \\ y_k^{(1)} = 0, \quad k = 2, \ldots, n \end{cases} \tag{97}
$$

with $\theta(T_0, T_1, T_2) = T_0 + \beta(T_1, T_2)$ for which we have $D_0\theta = 1,\ D_1\theta = D_1\beta_1$; we put terms involving $y_k^1,\ k \geq 2$ into R_k; so we obtain:

$$
\begin{aligned}
S_{2,1} = -\delta\phi_{1p}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big] \\
+ 2\omega_1(D_1 a_1 \sin(\theta) + a_1(D_1\beta_1 + \sigma)\cos(\theta)) + \lambda_1 a_1 \omega_1 \sin(\theta) \\
+ f_1(\cos(\theta)\cos(\beta_1) + \sin(\theta)\sin(\beta_1))
\end{aligned}
$$

$$
\begin{aligned}
S_{2,k} = -\delta\phi_{kp}\Big[\frac{ca_1^2}{2}(1 + \cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4}(\cos(3\theta) + 3\cos(\theta))\delta\phi_{1p}^3\Big] \\
+ f_k(\cos(\theta)\cos(\beta_1) + \sin(\theta)\sin(\beta_1)), \quad k = 1, \ldots, n.
\end{aligned}
$$

We will eliminate the terms at angular frequency ω_1 hence the functions $a_1(T_1, T_2)$ and $\beta_1(T_1, T_2)$ satisfy:

$$
\begin{cases} 2\omega_1 D_1 a_1 + \lambda_1 a_1 \omega_1 = -f_1 \sin(\beta_1) \\ 2\omega_1 a_1 D_1 \beta_1 + 2\omega_1 a\sigma - \frac{3d\delta\phi_{1p}^4 a_1^3}{4} = -f_1\cos(\beta_1) \end{cases} \tag{98}
$$

and the solution of the second equation of(96) is:

$$
\begin{cases} y_1^{(2)} = \delta\phi_{1p}\Big[(\frac{-ca_1^2}{2\omega_1^2} + \frac{ca_1^2}{6\omega_1^2}\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{32\omega_1^2}\cos(3\theta)\delta\phi_{1p}^3\Big] \\ y_k^{(2)} = \delta\phi_{kp}\Big[-\frac{ca_1^2}{2(\omega_k^2 - \omega_1^2)} + \frac{ca_1^2}{2(4\omega_1^2 - 2\omega_k^2)}\cos(2\theta))\delta\phi_{1p}^2 + \frac{da_1^3}{4(9\omega_1^2 - \omega_k^2)}\cos(3\theta)\delta\phi_{1p}^3\Big] \end{cases} \tag{99}
$$

where we have omitted the term at frequency ω_1 which is redundant with $y_1^{(1)}$

For the third equation of (96), the unknown is r_k; we do not solve it but we show that the solution is bounded on an interval dependent on ϵ. After some manipulations, the right hand side is:

$$S_{3,1} = +\sin\theta\left[2\omega_1 D_2 a_1 + \lambda_1 a_1 D_1\beta_1 + 2D_1 a_1 D_1\beta_1 + a_1 D_1^2\beta_1 + 2\sigma D_1 a_1 + \lambda_1 a_1\sigma\right]$$
$$+\cos\theta\left[2\omega_1 a_1 D_2\beta_1 - \lambda_1 D_1 a_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \sigma^2 a_1 + 2\sigma a_1 D_1\beta_1 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1} - \frac{3\delta\phi_{1p}^8 d^2 a_1^5}{128\omega_1}\right]$$
$$+ S_3^\sharp - \epsilon R(\epsilon, r, u^{(1)}, u^{(2)})$$

where

$$S_{3,1}^\sharp = \frac{5cd\delta\phi_{1p}^7 a_1^4}{8\omega_1^2} + \sin 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1}{3\omega_1} D_1 a_1 + \frac{\lambda_1 c\delta\phi_{1p}^3 a_1^2}{3\omega_1}\right]$$
$$+\cos 2\theta\left[\frac{4c\delta\phi_{1p}^3 a_1^2}{3\omega_1} D_1\beta_1 + \frac{15cd\delta\phi_{1p}^7 a_1^4}{32\omega_1^2}\right]$$
$$+\sin 3\theta\left[\frac{9d\delta\phi_{1p}^4 a_1^2}{16\omega_1} D_1 a_1 + \frac{3\lambda_1 d\delta\phi_{1p}^4 a_1^3}{16\omega_1}\right] + \cos 3\theta\left[\frac{9d\delta\phi_{1p}^4 a_1^3}{16\omega_1} D_1\beta_1 - \frac{c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3d^2\delta\phi_{1p}^8 a_1^4}{64\omega_1^2}\right]$$
$$+\cos 4\theta\left[-\frac{3cd\delta\phi_{1p}^7 a_1^4}{8\omega_1^2}\right] - \cos 5\theta\frac{3d^2\delta\phi_{1p}^8 a_1^5}{128\omega_1^2}$$

and a similar expression for $S_{3,k}^\sharp$. To eliminate the secular terms, we impose,

$$\begin{cases} 2\omega_1 D_2 a_1 + \lambda_1 a_1 D_1\beta_1 + 2D_1 a_1 D_1\beta_1 + a_1 D_1^2\beta_1 + 2\sigma D_1 a_1 + \lambda_1 a_1\sigma = 0 \\ 2\omega_1 a_1 D_2\beta_1 - \lambda_1 D_1 a_1 - D_1^2 a_1 + a_1(D_1\beta_1)^2 + \sigma^2 a_1 + 2\sigma a_1 D_1\beta_1 + \frac{5c^2\delta\phi_{1p}^6 a_1^3}{6\omega_1^2} - \frac{3\delta\phi_{1p}^8 d^2 a_1^5}{128\omega_1^2} = 0 \end{cases}$$

As a_1 and β_1 do not depend on T_0, the following relations hold:

$$\begin{cases} \dfrac{da_1}{dt} = \epsilon D_1 a_1 + \epsilon^2 D_2 a_1 + \mathcal{O}(\epsilon^3) \\ \dfrac{d\beta_1}{dt} = \epsilon D_1\beta_1 + \epsilon^2 D_2\beta_1 + \mathcal{O}(\epsilon^3) \end{cases} \tag{100}$$

On the other hand, we can determine the expression of $D_2 a_1$ and $D_2\beta$, like for one degree of freedom:

$$\begin{cases} D_2 a_1 = \frac{3d\lambda_1\delta\phi_{1p}^4 a_1^3}{16\omega_1^2} + \frac{\sigma f_1\sin\gamma}{4\omega_1^2} + \frac{\lambda_1 f_1\cos\gamma}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4 a_1^2 f_1\sin\gamma}{32\omega_1^3} \\ D_2\gamma = -\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} \\ \qquad + \frac{\sigma f_1\cos\gamma}{4\omega_1^2 a_1} + \frac{3d\delta\phi_{1p}^4 a_1 f_1\cos\gamma}{32\omega_1^3} - \frac{\lambda_1 f_1\sin\gamma}{8\omega_1^2 a_1} \end{cases} \tag{101}$$

now we return to (100) introducing (98) and (101), we obtain:

$$\frac{da_1}{dt} = \epsilon\Big(-\frac{f_1\sin(\beta)}{2\omega_1} + \frac{\lambda_1 a_1}{2}\Big) + \epsilon^2\left(\frac{3d\lambda_1\delta\phi_{1p}^4 a_1^3}{16\omega_1^2} + \frac{\sigma f_1\sin\beta}{4\omega_1^2} + \frac{\lambda_1 f_1\cos\beta}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4 a_1^2 f_1\sin\beta}{32\omega_1^3}\right) + O(\epsilon^3)$$
$$\frac{d\beta}{dt} = \epsilon\Big(-\sigma + \frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} - \frac{f_1\cos(\beta)}{2\omega_1 a_1}\Big)$$
$$+\epsilon^2\Big(-\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} + \frac{\sigma f_1\cos\beta}{4\omega_1^2 a_1} + \frac{3d\delta\phi_{1p}^4 a_1 f_1\cos\beta}{32\omega_1^3} - \frac{\lambda_1 f_1\sin\beta}{8\omega_1^2 a_1}\Big) + O(\epsilon^3) \tag{102}$$

Remark 18 In this approach, like for one free degree of freedom, we are using the method of reconstitution. We notice these equations are similar to (33)

Remark 19 $S_3^\sharp + R(\epsilon, r, u^{(1)}, u^{(2)})$ has no term at frequency ω_1 or which goes to ω_1 .
This will allow us to justify this expansion in certain conditions, before we consider the stationary solution of the system (102) and the stability of the solution close to the stationary solution.

3.2.2 Stationary solution and stability

Let us consider the stationary solution of (102), it satisfies:

$$\begin{cases} g_1(a_1, \beta_1, \sigma, \epsilon) = 0, \\ g_2(a_1, \beta_1, \sigma, \epsilon) = 0 \end{cases} \tag{103}$$

with

$$\begin{cases} g_1 = \epsilon(-\frac{f_1 \sin(\beta)}{2\omega_1} + \frac{\lambda_1 a_1}{2}) + \\ \qquad \epsilon^2(\frac{3d\lambda_1 \delta\phi_{1p}^4 a_1^3}{16\omega_1^2} + \frac{\sigma f_1 \sin\beta}{4\omega_1^2} + \frac{\lambda_1 f_1 \cos\beta}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4 a_1^2 f_1 \sin\beta}{32\omega_1^3}) + \mathcal{O}(\epsilon^3) \\ g_2 = \epsilon(-\sigma + \frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} - \frac{f_1 \cos(\beta)}{2\omega_1 a_1}) \\ \qquad + \epsilon^2(-\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} + \frac{\sigma f_1 \cos\beta}{4\omega_1^2 a_1} - \frac{3d\delta\phi_{1p}^4 a_1 f_1 \cos\beta}{32\omega_1^3} - \frac{\lambda_1 f_1 \sin\beta}{8\omega_1^2 a_1}) + \mathcal{O}(\epsilon^3) \end{cases} \tag{104}$$

The situation is very close to the 1 d.o.f. case; except the replacement of c by $\check{c} = c\delta\phi_{1p}^3$ and d by of $\check{d} = d\delta\phi_{1p}^4$, the system (104) is the same as (35); the other components are zero. We state a similar proposition.

Proposition 6 *When*

$$\sigma \leq \frac{3\check{d}\bar{a}_1^2}{4\omega_1} - \frac{1}{2}\sqrt{\frac{9\check{d}^2\bar{a}_1^4}{16\omega_1^2} - \lambda_1^2}$$

and ϵ small enough, the stationary solution $(\bar{a}_1, \bar{\beta}_1)$ of (102) *is stable in the sense of Lyapunov (if the dynamic solution starts close to the stationary one, it remains close and converges to it); to the stationary case corresponds the approximate solution of* (91)

$$\tilde{y}_{1\,app} = \epsilon\bar{a}_1 \cos(\tilde{\omega}_\epsilon t + \bar{\beta}) + \epsilon^2\Big(\delta\phi_{1p}\Big[\Big(\frac{-c\bar{a}_1^2}{\omega_1^2} + \frac{c\bar{a}_1^2}{6\omega_1^2}\cos 2(\tilde{\omega}_\epsilon t + \bar{\beta})\Big)\delta\phi_{1p}^2 + \frac{d\bar{a}_1^3}{32\omega_1^2}\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}))\delta\phi_{1p}^3\Big]\Big); \tag{105}$$

$$\begin{aligned} \tilde{y}_{k\,app} = \epsilon^2\Big(\delta\phi_{kp}\Big[\Big(\frac{-c\bar{a}_1^2}{2(\tilde{\omega}_k^2 + \omega_1^2)} - \frac{c\bar{a}_1^2}{2(\omega_k^2 - 4\omega_1^2)}\cos 2(\tilde{\omega}_\epsilon t + \bar{\beta})\Big)\delta\phi_{1p}^2 \\ - \frac{d\bar{a}_1^3}{4(\tilde{\omega}_k^2 - 9\omega_1^2)}\cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}))\delta\phi_{1p}^3\Big]\Big) \end{aligned} \tag{106}$$

it is periodic.

With this result of stability, we can state precisely the approximation of the solution of (88)

3.2.3 Convergence of the expansion

In order to prove that r_k is bounded, after eliminating terms at frequency ν_1, we go back to the variable t for the third set of equations of (96) .

$$\begin{aligned}
&\frac{d^2 r_k}{dt^2} + \omega_1^2 r_k = \tilde{S}_{3,k} \quad \text{for } k = 1, \dots n \quad \text{with} \\
&\tilde{S}_{3,1} = S^\sharp_{3,1}(t,\epsilon) - \epsilon \tilde{R}_1(y_1^{(1)}, y_1^{(2)}, r_1, \epsilon) \quad \text{and for } k \neq 1 \\
&S_{3,k} = -2c\delta\phi_{kp}[y_1^{(1)} y_1^{(2)} \delta\phi_{1p}^2] - 3d\delta\phi_{kp}[y_1^{(1)2} y_1^{(2)} \delta\phi_{1p}^3] - \epsilon R_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)})
\end{aligned}$$

where

$$\tilde{R}_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)}) = R_k(\epsilon, r_k, y_1^{(1)}, y_1^{(2)}) - \mathcal{D}_2 r_k - \lambda_k\big(\frac{dr_k}{dt} - \omega_k D_0 r_k\big)$$

with all the terms expressed with the variable t.

Proposition 7 *Under the assumption that $\omega_k^2 \neq 4\omega_1^2$, $\omega_k^2 \neq 9\omega_1^2$ and ω_1^2 a simple eigenvalue (no internal resonance) for $k \neq 1$, there exists $\varsigma > 0$ such that for all $t \leq t_\epsilon = \frac{\varsigma}{\epsilon^2}$, the solution $\tilde{y} = \epsilon y$ of* (90) *with initial data*

$$\begin{aligned}
\tilde{y}_1(0) &= \epsilon a_1 + \epsilon^2 \,\big(\frac{-\check{c}_1 a_{10}^2}{2\omega_1^2} \;+\; \frac{\check{c}_1 a_{10}^2}{6\omega_1^2}\cos(2\beta_0) \;+\frac{\check{d}_1 a_{10}^3}{32\omega_1^2}\cos(3\beta_0)\big) + \epsilon^3 r(0,\epsilon), \\
\tilde{y}_k(0) &= \epsilon^2 \left(\frac{-\check{c}_1 a_{10}^2}{2(\omega_k^2 - \omega_1^2)} + \frac{\check{c}_1 a_{10}^2}{2(4\omega_1^2 - \omega_k^2)}\cos(2\beta_0) + \frac{\check{d}_1 a_{10}^3}{4(9\omega_1^2 - \omega_k^2)}\cos(3(\beta_0)) \right) + \epsilon^3 r(0,\epsilon),
\end{aligned}$$

with similar expressions for $\dot{y}_1(0), \dot{y}_k(0)$ and with (a_{10}, β_0) close to the stationary solution $(\bar{a}_1, \bar{\beta})$

$$|a_{10} - \bar{a}_1| \leq \epsilon^2 C^1, \quad |\beta_0 - \bar{\beta}| \leq \epsilon^2 C^1$$

has the following expansion

$$\tilde{y}_1 = \epsilon a_1 \cos(\tilde{\omega}_\epsilon t + \beta(t)) + \epsilon^2[((\frac{-\check{c}_1 a_1^2}{2\omega_1^2} + \frac{\check{c}_1 a_1^2}{6\omega_1^2}\cos(2(\tilde{\omega}_\epsilon t + \beta(t))) + \frac{\check{d}_1 a_1^3}{32\omega_1^2}\cos(3(\tilde{\omega}_\epsilon t + \beta(t))))] + \epsilon^3 r_1(t)$$

$$\tilde{y}_k = \epsilon^2\Big(\Big[\big(\frac{-\check{c}_k a_1^2}{2(\omega_k^2 - \omega_1^2)} + \frac{\check{c}_k a_1^2}{2(4\omega_1^2 - \omega_k^2)}\cos(2(\tilde{\omega}_\epsilon t + \beta(t))) + \frac{\check{d}_k a_1^3}{4(9\omega_1^2 - \omega_k^2)}\cos(3(\tilde{\omega}_\epsilon t + \beta(t)))\Big]\Big) + \epsilon^3 r_k(t)$$

with a_1, β solution of (102) *and with r_k uniformly bounded in $\mathcal{C}^2(0, t_\epsilon)$ for $k = 1, \dots n$ and ω_1, ϕ_1 are the eigenvalue and eigenvectors defined in* (65)*, with $\delta\phi_{1p} = (\phi_{1,p} - \phi_{1,p-1}), \delta\phi_{kp} = (\phi_{k,p} - \phi_{k,p-1})$, $\check{c}_1 = c(\delta\phi_{1p})^3$, $\check{d}_1 = d(\delta\phi_{1p})^4$ and $\check{c}_k = c(\delta\phi_{1p})^2\delta\phi_{kp}$, $\check{d}_k = d(\delta\phi_{1p})^3\delta\phi_{kp}$ as in proposition 5.*

Corollary 2 *The solution of* (88) *with*

$$\begin{aligned}
\phi_1^T \tilde{u}(0) &= \epsilon a_1 + \epsilon^2\big(\frac{-\check{c}_1 a_{10}^2}{2\omega_{10}^2} \;+\; \frac{-\check{c}_1 a_{10}^2}{6\omega_1^2}\cos(2\gamma_0) \;+\frac{\check{d}_1 a_{10}^3}{32\omega_1^2}\cos(3\gamma_0)\big) + \epsilon^3 r_1(0,\epsilon), \\
\phi_k^T \tilde{u}(0) &= \epsilon^2 \left(\frac{-\check{c}_1 a_{10}^2}{2(\omega_k^2 - \omega_1^2)} + \frac{\check{c}_1 a_{10}^2}{2(4\omega_1^2 - \omega_k^2)}\cos(2\gamma_0) + \frac{\check{d}_1 a_{10}^3}{4(9\omega_1^2 - \omega_k^2)}\cos(3(\gamma_0)) \right) + \epsilon^3 r_k(0,\epsilon),
\end{aligned}$$

with similar expressions for $\phi_1^T \dot{\tilde{u}}(0), \phi_k^T \dot{\tilde{u}}(0)$ *and with* ω_k, ϕ_k *the eigenvalues and eigenvectors defined in* (65).

$$\text{is } \tilde{u}(t) = \sum_{k=1}^{n} \tilde{y}_k(t)\phi_k \tag{107}$$

with the expansion of y_k *of previous proposition.*

Proof We follow a similar route as for one degree of freedom, we use lemma 4. Set $S_1 = S_{31}^{\sharp}$, $S_k = S_{3,k}$ for $k = 1, \dots n$; as we have enforced (104), the functions S_k are not periodic but close to a periodic function, bounded and are orthogonal to $e^{\pm it}$, we have assumed that ω_k and ω_1 are $\mathbb{Z}$ independent for $k \neq 1$; so S satisfies the lemma hypothesis. Similarly, set $g = \tilde{R}$, it is a polynomial in r with coefficients which are bounded functions , so it is lipschitzian on the bounded subsets of $\mathbb{R}$, it satisfies the hypothesis of lemma 4 and so the proposition is proved. The corollary is an easy consequence of the proposition and the change of function (89) □

3.2.4 Maximum of the stationary solution

We can state results similar to the case of one degree of freedom.

Proposition 8 *The stationary solution of* (102) *satisfies*

$$\begin{cases} (-\frac{f_1 \sin(\beta)}{2\omega_1} + \frac{\lambda_1 a_1}{2}) + \qquad \epsilon A_1(a_1, \beta, \sigma) + \mathcal{O}(\epsilon^2) = 0 \\ (-\sigma + \frac{3d\delta\phi_{1p}^4 a_1^2}{8\omega_1} - \frac{f_1 \cos(\beta)}{2a_1\omega_1}) + \qquad \epsilon A_2(a_1, \beta, \sigma) + \mathcal{O}(\epsilon^2) = 0 \end{cases} \tag{108}$$

with

$$A_1(a, \beta, \sigma) = \frac{3d\delta\phi_{1p}^4 \lambda_1 a_1^3}{16\omega_1^2} + \frac{\sigma f_1 \sin\beta}{4\omega_1^2} + \frac{\lambda f_1 \cos\beta}{8\omega_1^2} + \frac{9d\delta\phi_{1p}^4 a_1^2 f_1 \sin\beta}{32\omega_1^3}$$

$$A_2(a, \beta, \sigma) = -\frac{\lambda_1^2}{8\omega_1} - \frac{15d^2\delta\phi_{1p}^8 a_1^4}{256\omega_1^3} - \frac{5c^2\delta\phi_{1p}^6 a_1^2}{12\omega_1^3} + \frac{\sigma f_1 \cos\beta}{4\omega_1^2 a_1} + \frac{3d\delta\phi_{1p}^4 a_1 f_1 \cos\beta}{32\omega_1^3} - \frac{\lambda_1 f_1 \sin\beta}{8\omega_1^2 a_1}$$

this stationary solution reaches its maximum amplitude for $\sigma = \sigma_0^* + \epsilon\sigma_1^* + O(\epsilon^2)$ *with*

$$a_{1,0}^* = \frac{f_1}{\lambda_1\omega_1}, \; \sigma_0^* = \frac{3\check{d}a_{1,0}^{*2}}{8\omega_1} = \frac{3\check{d}f_1^2}{8\lambda_1^2\omega_1^3}, \quad \beta_0^* = -\frac{\pi}{2} \tag{109}$$

and

$$\sigma_1^* = -\frac{87\check{d}^2 a_{1,0}^{*4}}{256\omega_1^3} - \frac{5\check{c}^2 a_{1,0}^{*2}}{12\omega_1^3} - \frac{\lambda_1^2}{4\omega_1}, \quad \beta_1^* = -\frac{\lambda_1}{2\omega_1}, \quad a_{1,1}^* = -\frac{a_{1,0}^*\sigma_0^*}{\omega_1}$$

the periodic forcing is at the angular frequency

$$\tilde{\omega}_\epsilon = \omega_1 + \epsilon\sigma_0^* + \epsilon^2\sigma_1^* + \mathcal{O}(\epsilon^2)$$

up to the term involving the damping ratio λ_1, it is slightly different of the approximate angular frequency ν_ϵ of the undamped free periodic solution (85)*; for this frequency, the approximation (of the solution $\tilde{y} = \epsilon y$ of* (90) *up to the order ϵ^2) is periodic:*

$$\begin{cases} \tilde{y}_1(t) = \epsilon \bar{a}_1^* \cos(\tilde{\omega}_\epsilon t + \bar{\beta}^*) + \epsilon^2 [(\frac{-\check{c}_1 \bar{a}_1^{*2}}{2\omega_1^2} + \frac{\check{c}_1 \bar{a}_1^{*2}}{6\omega_1^2} \cos(2(\tilde{\omega}_\epsilon t + \bar{\beta}^*))) \\ \qquad\qquad + \frac{\check{d}_1 \bar{a}_1^{*3}}{32\omega_1^2} \cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}^*)] + \epsilon^3 r_1(\epsilon, t) \\ \tilde{y}_k(t) = \epsilon^2 [(\frac{-\check{c}_k \bar{a}_1^{*2}}{2(\omega_k^2 - \omega_1^2)} - \frac{\check{c}_k \bar{a}_1^{*2}}{2(\omega_k^2 - 4\omega_1^2)} \cos(2(\tilde{\omega}_\epsilon t + \bar{\beta}^*))) \\ \qquad\qquad - \frac{\check{d}_k \bar{a}_1^{*3}}{4(\omega_k^2 - 9\omega_1^2)} \cos(3(\tilde{\omega}_\epsilon t + \bar{\beta}^*)] + \epsilon^3 r_k(\epsilon, t) \end{cases} \tag{110}$$

and initial conditions like in proposition 5.

4 Conclusion

For some differential systems modelling spring-masses vibrations with non linear springs, we have derived and rigorously proved an asymptotic approximation of periodic solution of free vibrations (so called non linear normal modes); for damped vibrations with periodic forcing with frequency close (but different) to free vibration frequency (the so called primary resonance case), we have obtained an asymptotic expansion and derived that the amplitude is maximal close to the frequency of the non linear normal mode.

We emphasize that the use of three time scales provides a more accurate value of the link between frequency and amplitude (so called backbone) of a non linear mode but it yields also a new insight in the behavior of the solution which was not provided by a double-scale analysis: the influence of the ratio of c over d on the shape of the backbone and the amplitude of the forced response to an harmonic force as is clearly displayed in figure 2 and 3.

As an opening to a related problem, we can notice that such non linear vibrating systems linked to a bar generate acoustic waves; an analysis of the dilatation of a one-dimensional nonlinear crack impacted by a periodic elastic wave, with a smooth model of the crack may be carried over with a delay differential equation, [JL09].

Acknowledgment We thank S. Junca for his stimulating interest.

5 Appendix

All these lemmas are recalled here for convenience of the reader; they already have been proposed in [BR13].

Lemma 1 *Let w_ϵ be solution of*

$$\begin{aligned} w" + w &= S(t, \epsilon) + \epsilon g(t, w, \epsilon) \\ w(0) = 0, &\quad w'(0) = 0 \end{aligned} \tag{111}$$

If the right hand side satisfies the following conditions

1. *S is a sum of periodic bounded functions:*
   (a) *for all t and for all ϵ small enough, $S(t, \epsilon) \le M$*
   (b) *$\int_0^{2\pi} e^{it} S(t, \epsilon) dt = 0, \quad \int_0^{2\pi} e^{-it} S(t, \epsilon) dt = 0$ uniformly for ϵ small enough*

2. *for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the inequality $|g(t,u,\epsilon) - g(t,v,\epsilon)| \leq k_R|u-v|$ holds and $|g(t,0,\epsilon)|$is bounded; in other words g is locally lipschitzian with respect to u.*

then, there exists $\gamma > 0$ such that for ϵ small enough, w_ϵ is uniformly bounded in $C^2(0,T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof The proof is close to the proof of lemma 6.3 of [JR10]; but it is technically simpler since here we assume g to be locally lipschitzian with respect to u whereas it is only bounded in [JR10].

1. We first consider

$$\begin{aligned} w_1" + w_1 &= S(t,\epsilon) \\ w_1(0) = 0, \quad w_1'(0) &= 0 \end{aligned} \tag{112}$$

as S is a sum of periodic functions which are uniformly orthogonal to e^{it} and e^{-it}, w_1 is bounded in $\mathcal{C}^2(0,+\infty)$

2. Then we perform a change of function: $w = w_1 + w_2$, the following equalities hold

$$\begin{aligned} w_2" + w_2 &= \epsilon g_2(t, w_2, \epsilon) \\ w_2(0) = 0, \quad w_2'(0) &= 0 \end{aligned} \tag{113}$$

with g_2 which satisfies the same hypothesis as g:
for all $R > 0$, there exists k_R such that for $|u| \leq R$ and $|v| \leq R$, the following inequality holds $|g_2(t,u,\epsilon) - g_2(t,v,\epsilon)| \leq k_R|u-v|$. Using Duhamel principle, the solution of this equation satisfies:

$$w_2 = \epsilon \int_0^t \sin(t-s) g_2(s, w_2(s), \epsilon) ds$$

from which

$$|w_2(t)| \leq \epsilon \int_0^t |g_2(s, w_2(s), \epsilon) - g_2(s,0,\epsilon)| ds + \epsilon \int_0^t |g_2(s,0,\epsilon)| ds$$

so if $|w| \leq R$, hypothesis of lemma imply

$$|w_2(t)| \leq \epsilon \int_0^t k_R |w_2| ds + \epsilon C t$$

A corollary of lemma of Bellman-Gronwall, see below, will enable to conclude. It yields

$$|w_2(t)| \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1 \right)$$

Now set $T_\epsilon = \sup\{t \| w| \leq R\}$, then we have

$$R \leq \frac{C}{k_R} \left(\exp(\epsilon k_R t) - 1 \right)$$

this shows that there exists γ such that $|w_2| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0,T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have w in $\mathcal{C}(0,T_\epsilon)$ then as w is solution of (111), it is also bounded in $\mathcal{C}^2(0,T_\epsilon)$.

□

Lemma 2 *(Bellman-Gronwall, [BG, Bel64]) Let u, ϵ, β be continuous functions with $\beta \geq 0$,*

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s)u(s)ds \text{ for } 0 \leq t \leq T$$

then

$$u(t) \leq \epsilon(t) + \int_0^t \beta(s)\epsilon(s)\left[\exp(\int_s^t \beta(\tau)d\tau\right] ds$$

Lemma 3 *(a consequence of previous lemma, suited for expansions, see [SV85]) Let u be a positive function, $\delta_2 \geq 0$, $\delta_1 > 0$ and*

$$u(t) \leq \delta_2 t + \delta_1 \int_0^t u(s)ds$$

then

$$u(t) \leq \frac{\delta_2}{\delta_1}\left(exp(\delta_1 t) - 1\right)$$

Lemma 4 *Let $v_\epsilon = [v_1^\epsilon, \dots, v_N^\epsilon]^T$ be the solution of the following system:*

$$\omega^2(v_k^\epsilon)" + \omega_k^2 v_k^\epsilon = S_k(t) + \epsilon g_k(t, v_\epsilon) \tag{114}$$

If ω and ω_k are $\mathbb{Z}$ independent for all $k = 2 \dots N$ and the right hand side satisfies the following conditions with $M > 0$, $C > 0$ prescribed constants:

1. *S_k is a sum of bounded periodic functions, $|S_k(t)| \leq M$ which satisfy the non resonance conditions:*
2. *S_1 is orthogonal to $e^{\pm it}$, i.e. $\int_0^{2\pi} S_1(t)e^{\pm it}dt = 0$ uniformly for ϵ going to zero*
3. *for all $R > 0$ there exists k_R such that for $\|u\| \leq R$, $\|v\| \leq R$, the following inequality holds for $k = 1, \dots, N$:*
$$|g_k(t, u, \epsilon) - g_k(t, v, \epsilon)| \leq k_R\|u - v\|$$
and $|g_k(t, 0, \epsilon)|$ is bounded

then there exists $\gamma > 0$ such that for ϵ small enough v_ϵ is bounded in $\mathcal{C}^2(0, T_\epsilon)$ with $T_\epsilon = \frac{\gamma}{\epsilon}$

Proof

1. We first consider the linear system
$$\begin{aligned} \omega_1^2(v_{k,1})" + \omega_k^2 v_{k,1} &= S_k \\ v_{k,1}(0) = 0 \text{ and } (v_{k,1})' &= 0 \end{aligned} \tag{115}$$
For $k = 1$, with hypothesis 1.a, S_1 is a sum of bounded periodic functions; it is orthogonal to $e^{\pm it}$, there is no resonance. For $k \neq 1$, there is no resonance as $\frac{\omega_k}{\omega_1} \notin \mathbb{Z}$ with hypothesis 1.b. So $v_{k,1}$ belongs to $C^{(2)}$ for $k = 1, ..., n$
2. Then we perform a change of function
$$v_k^\epsilon = v_{k,1} + v_{k,2}^\epsilon$$
and $v_{k,2}^\epsilon$ are solutions of the following system :
$$\begin{aligned} \omega_1^2(v_{k,2})" + \omega_k^2 v_{k,2} &= \epsilon g_{k,2}(t, v_{k,2}, \epsilon),\ k = 1, \dots, N \\ v_{k,2}^\epsilon(0) = 0,\ (v_{k,2}^\epsilon)' &= 0,\ k = 1, \dots, N \end{aligned} \tag{116}$$

with

$$g_{k,2}(t, ..., v^{\epsilon}_{k,2}, ...) = g_k(t, ..., v_{k,1} + v^{\epsilon}_{k,2}, ...)$$

where $g_{k,2}$ satisfies the same hypothesis as g_k:
for all $R > 0$ there exists k_R such that for $\| u_k \| \leq R$, $\| v_k \| \leq R$, the following inequality holds for $k = 1, \ldots, N$:

$$\| g_{k,2}(t, u_k, \epsilon) - g_{k,2}(t, v_k, \epsilon) \| \leq k_R \| u_k - v_k \| \tag{117}$$

Using Duhamel principle, the solution or the equation (116) satisfies:

$$v^{\epsilon}_{k,2} = \epsilon \int_0^t \sin(t-s) g_{k,2}(s, v^{\epsilon}_{k,2}(s), \epsilon) ds$$

so

$$\| v^{\epsilon}_{k,2}(t) \| \leq \epsilon \int_0^t \| g_{k,2}(s, v^{\epsilon}_{k,2}(s), \epsilon) - g_{k,2}(s, 0, \epsilon) \| ds + \\ \epsilon \int_0^t \| g_{k,2}(s, 0, \epsilon) \| ds$$

so with (117), we obtain

$$\| v^{\epsilon}_{k,2}(t) \| \leq \epsilon \int_0^t k \| v^{\epsilon}_{k,2}(t) \| ds + \epsilon C t$$

We shall conclude using Bellman-Gronwall lemma; we obtain

$$\| v_{k,2}(t) \| \leq \frac{C}{k_R}(exp(\epsilon k_R t) - 1)$$

this shows that there exists γ such that $|v^{\epsilon}_{k,2}| \leq R$ for $t \leq T_\epsilon$, which means that it is in $L^\infty(0, T_\epsilon)$ for $T_\epsilon = \frac{\gamma}{\epsilon}$; also, we have v_k in $\mathcal{C}(0, T_\epsilon)$ then as v_k is solution of (111), it is also bounded in $\mathcal{C}^2(0, T_\epsilon)$.

Theorem 1 *(of Poincaré-Lyapunov, for example see [SV85]) Consider the equation*

$$\dot{x} = (A + B(t))x + g(t, x), \ x(t_0) = x_0, \ t \geq t_0$$

where $x, x_0 \in \mathbf{R}^n$, A is a constant matrix $n \times n$ with all its eigenvalues with negative real parts; $B(t)$ is a matrix which is continuous with the property $\lim_{t \to +\infty} \|B(t)\| = 0$. *The vector field is continuous with respect to t and x is continuously differentiable with respect to x in a neighbourhood of $x = 0$; moreover*

$$g(t, x) = o(\|x\|) \ when \ \|x\| \to 0$$

uniformly in t. Then, there exists constants C, t_0, δ, μ such that if $\|x_0\| < \frac{\delta}{C}$

$$\|x\| \leq C \|x_0\| e^{-\mu(t-t_0)}, t \geq t_0$$

holds

5.1 Numerical computations of Fourier transform

Assuming a function f to be almost-periodic, the Fourier coefficients are :

$$\alpha_n = \lim_{T\to+\infty} \int_0^T f(t)e^{-i\lambda_n t}dt \tag{118}$$

where λ_n are countable Fourier exponents of f. (for example, see Fourier coefficients of an almost-periodic function in http://www.encyclopediaofmath.org/). For numerical purposes, we chose T large enough and with a fast Fourier transform, we compute numerically the Fourier coefficients of a function of period T equal to f in this interval.

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2.2 Autres résultats numériques

Le cas de deux éléments dont un endomagé refléte le système, étant donné que la barre est saine par tout sauf en une partie.
les solutions sont :

$$\begin{cases} y_1(t) & = \epsilon a_1 \cos(\nu_\epsilon t) + \epsilon^2[-C_1 + C_2 cos(2(\nu_\epsilon t)) + D cos(3(\nu_\epsilon t))] \\ & \quad + \epsilon^3 r_1(\epsilon, t) \\ y_2(t) & = \epsilon^2[-C_{12} - C_{22} cos(2(\nu_\epsilon t)) - D_2 cos(3(\nu_\epsilon t))] + \epsilon^3 r_2(\epsilon, t) \end{cases}$$

tel que $\delta_{12} = (\phi_{1,2} - \phi_{1,1})$
avec
$C_1 = \dfrac{c(\delta\phi_{12})^3 a_1^2}{2\omega_1^2}$, $C_2 = \dfrac{c(\delta\phi_{12})^3 a_1^2}{6\omega_1^2}$, $D = \dfrac{d(\delta\phi_{12})^4 a_1^3}{32\omega_1^2}$,
$C_{12} = \dfrac{c\delta\phi_{22}(\delta\phi_{12})^2 a_1^2}{2\omega_2^2}$, $C_{22} = \dfrac{c\delta\phi_{22}(\delta\phi_{12})^2 a_1^2}{2(\omega_2^2 - 4\omega_1^2)}$, $D_2 = \dfrac{d(\delta\phi_{12})^3 \delta\phi_{22} a_1^3}{4(9\omega_2^2 - \omega_1^2)}$ On note DA : développement asymptotique.

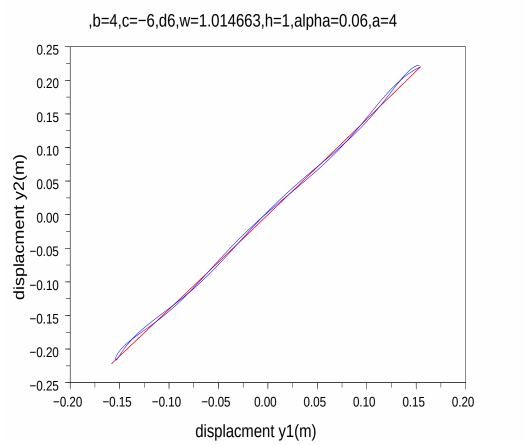

FIG. 2.1 – Déplacement temporel non linéaire D.A (rouge) , ODE(bleu)

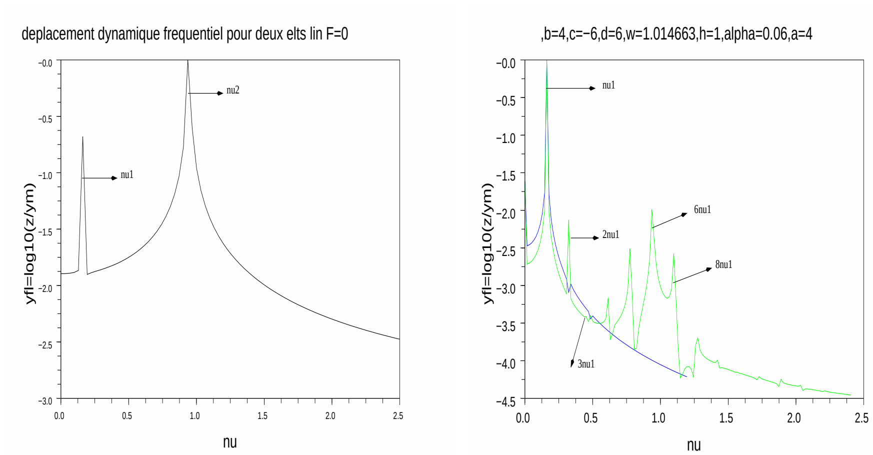

FIG. 2.2 – (à gauche) : Déplacement dynamique fréquentiel (fft) linéaire - (à droite) non linéaire DA(bleu) et ODE(vert) pour 2 éléments,z est la norme de la fft(y_1 où y_2)

Le problème pour 2 élèments a été résolu avec deux méthodes : développement asymptotique et résolution numérique, où nous avons utilisé pour ce dernier, le programme ODE. Ces deux méthodes nous ont permi d'obtenir une solution **u** que nous avons utilisé pour dessiner une courbe paramétrée de ces deux premières composantes ; $y_1 = u(1,:)$ et $y_2 = u(2.:)$ en imposant une condition initiale pour obtenir une solution périodique selon le principe des modes normaux non linéaire ; avec le développement asymptotique dans la figure 2.1 en rouge, on trouve une solution proche du segment de droite obtenu dans le cas linéaire ; avec la solution numérique dans la figure 2.1 en bleu, nous trouvons une courbe ayant la forme d'un haricot dû au comportement des composantes en fonction du temps et au choix de la condition initiale ; nous remarquons aussi quelques écartements dû aux manque de précision de la condition initiale, ensuite, nous avons calculé **la transformée de Fourier** de y_1 pour avoir la réponse fréquentielle, nous avons dessiné log10(z/ym) (où z est la norme de la fft(y1) et ym est le max de (z)) en fonction de la fréquence. Pour le système linéaire, il y a deux fréquences qui sont : $\nu_1 = 0.1614805$ Hz et $\nu_2 = 0.9411773$ Hz et pour le cas non linéaire, en choisissant cette condition initiale, il n'y a que les fréquences de période multiple de la première fréquence non linéaire qui est proche de ν_1 représentées par la figure 2.2, c'est le principe des modes normaux.

En remarquant l'importance des termes cubiques et quadratiques, nous avons procédé à des expériences où nous avons varié les coefficients c et d qui sont représentées par les figures suivantes : .

▶**Variation de c**

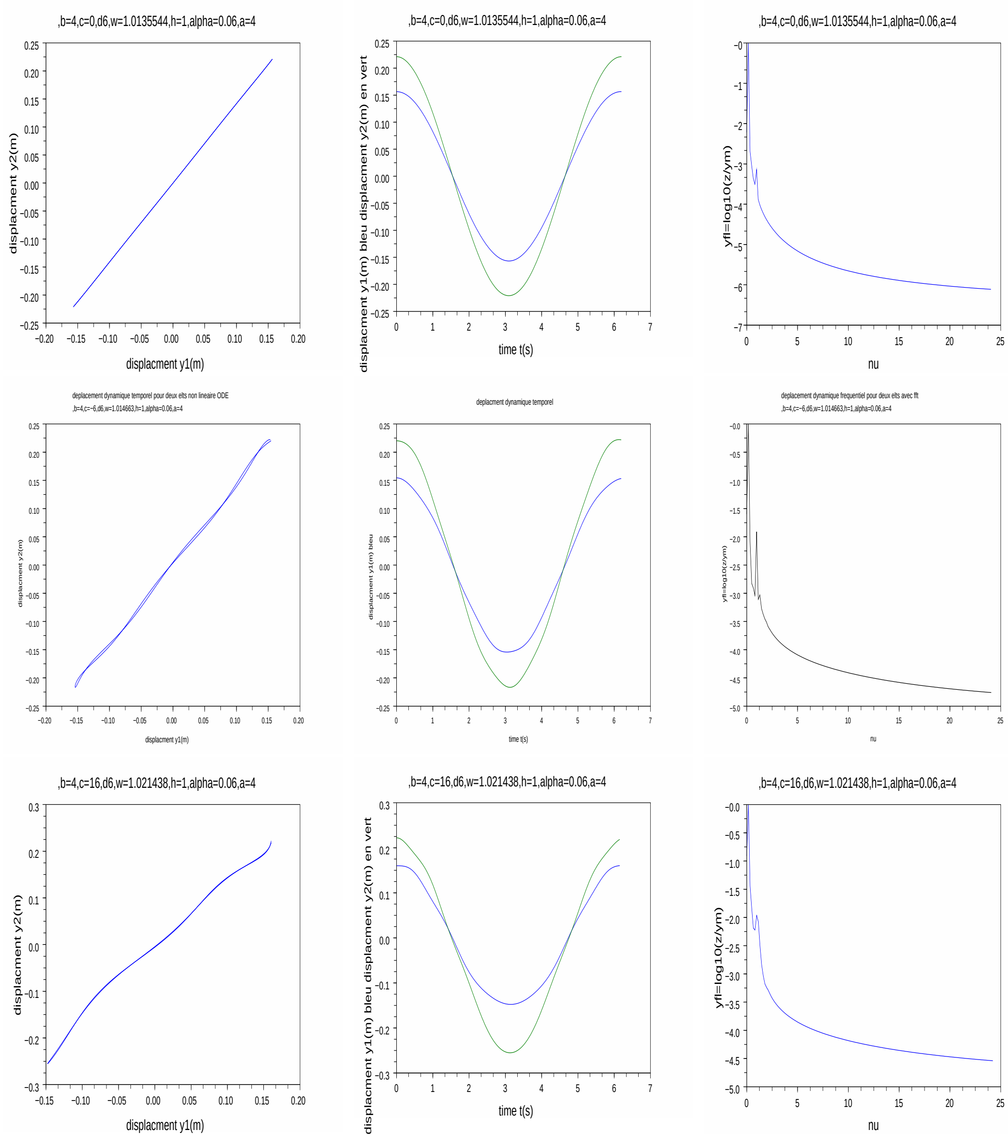

FIG. 2.3 – Différentes courbes pour différentes valeurs de la variable quadratique 'c' (resp c=0, c=-6, c=16) de la non linéarité titre avec le programme ODE

Lorsque nous avons enlevé la partie quadratique c'est à dire c=0, nous nous rapprochons du cas linéaire, il ne reste que les fréquences $3\nu_1$ et $6\nu_1$ multiple de $3\nu_1$ mais en petite amplitude.
En augmantant la valeur de c, nous remarquons que ces courbes reprennent leurs formes mais avec des ondulations plus importantes, le plus important, c'est de differentier entre l'absence de la partie quadratique et sa présence et le changement qu'il y a eu .

▶**Variation de d** :

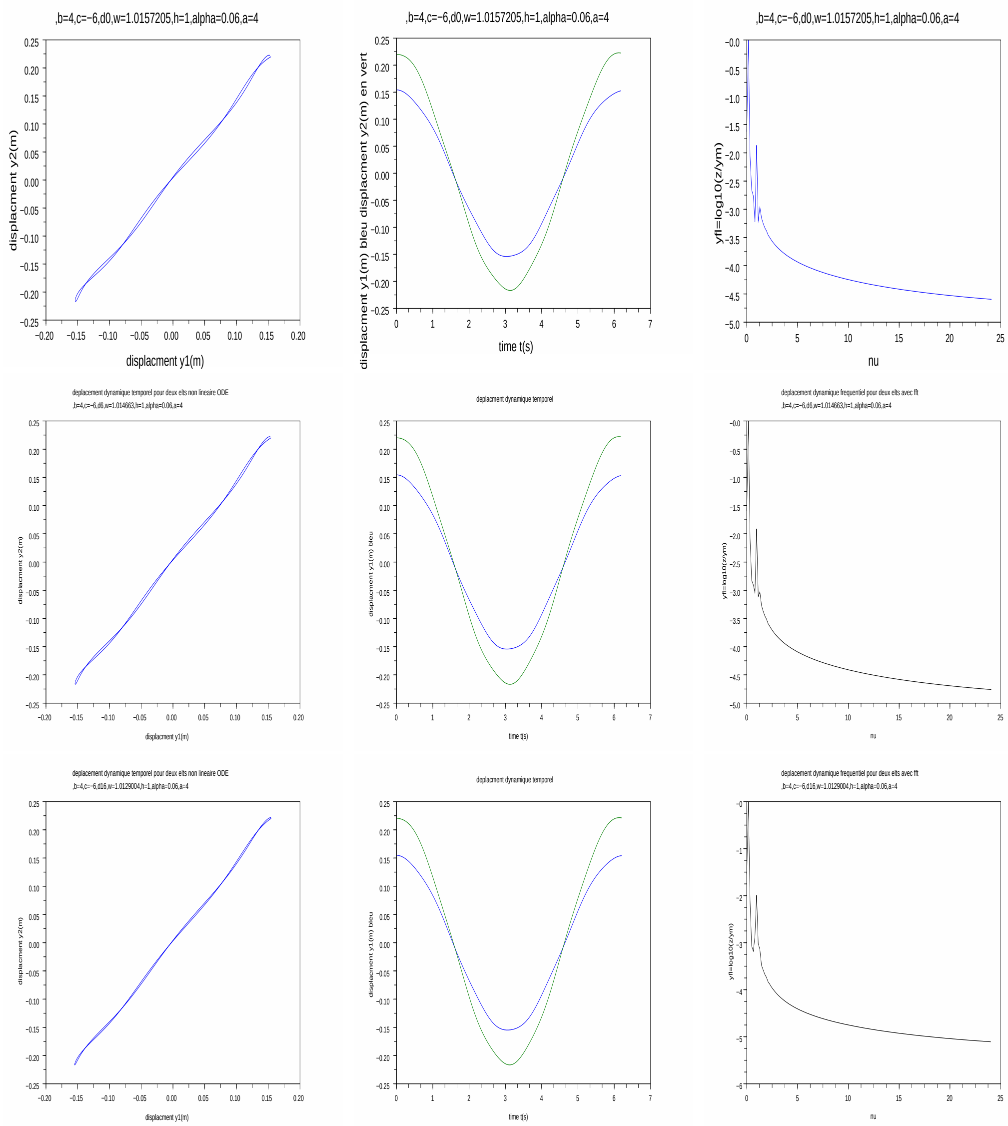

FIG. 2.4 – Différentes courbes pour différentes valeurs de la variable cubique 'd'(resp. d=0, d=6, d=-16) de la non linéarité avec le programme ODE

Lorsque nous avons éliminé d, l'allure de la courbe paramétrée n 'a pas beaucoup changé cela est dû au fait que la partie cubique ne contient pas seulement les termes en d mais aussi des termes en c.

Ce qui nous a amené à varier la variable ϵ

▶**Variation d'**ϵ :

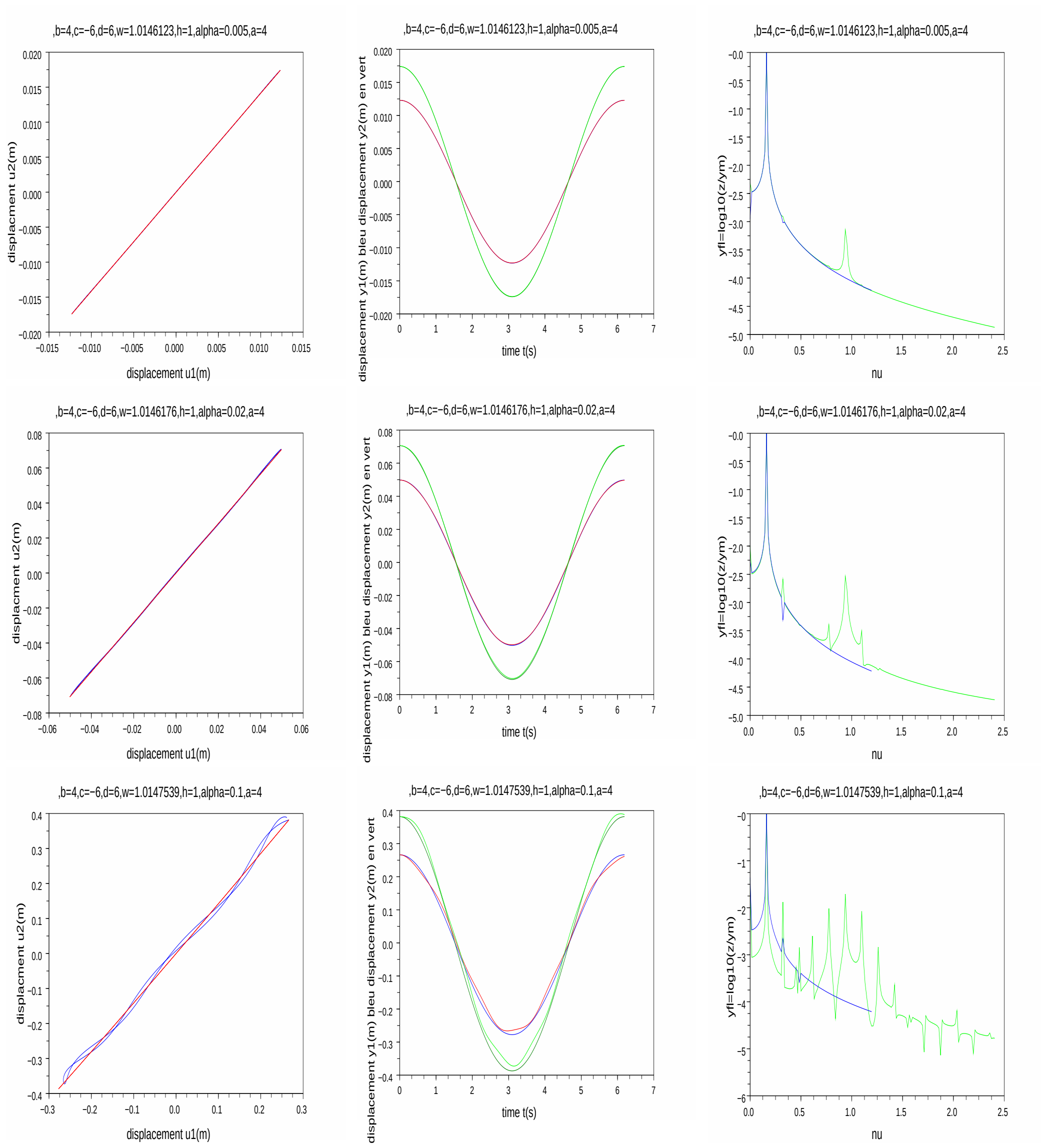

FIG. 2.5 – Différents courbes pour différents valeurs du paramétre perturbateur ϵ(resp $\epsilon = 0.005, \epsilon = 0.02, \epsilon = 0.1$) avec le DA et ODE

En augmentant la valeur d'ϵ , nous remarquons que les ondulations sont plus importantes alors nous déduisons que les développements asymptotiques ne sont fiables que pour des valeurs d'ϵ trés petites, en effet, c'est le principe de ces développements.

Chapitre 3

Présentation d'un modèle Elasto-plastique unifié

3.1 Introduction

Un matériau élastoplastique est caractérisé par un seuil d'effort au-dessous duquel le comportement est purement élastique (réversible). Une fois celui-ci atteint, la déformation n'est plus réversible. Généralement, la déformation est considérée indépendante du temps et elle n'évolue pas si la charge est maintenue constante (pas de viscosité et pas de fluage). Une fois le seuil, qui n'est pas fonction de la vitesse de la déformation, est dépassé, des défauts localisés peuvent apparaître provoquant une diminution brutale de la rigidité. Après décharge, le comportement redevient élastique, tout en gardant une déformation résiduelle mais ayant une limite d'élasticité plus importante qu'avant le dépassement du seuil.
Le comportement du matériau est dit élasto-plastique avec écrouissage. La courbe d'écrouissage est la courbe de l'évolution de la contrainte σ en fonction de l'allongement ϵ représenté par la figure 3.1.

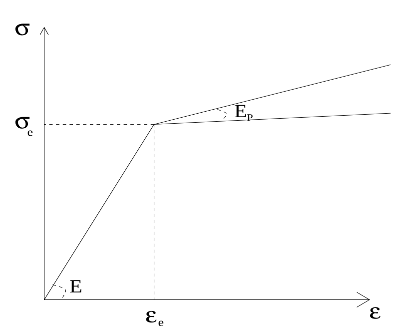

FIG. 3.1 – Modèle élastoplastique avec Ecrouissage

Dans la première partie, $(\sigma < \sigma_e)$, le comportement est dit élastique.
Dans la deuxième partie,$(\sigma > \sigma_e)$, la rigidité diminue considèrablement $(E_p << E)$

S'il y a une décharge, le comportement du matériau redevient élastique avec la rigidité initiale E, figure 3.2.
S'il y a une décharge complète, la déformation à $\sigma = 0$ devient ϵ_0 (c'est la déformation plastique résiduelle).

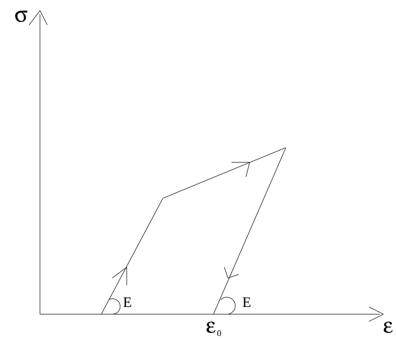

FIG. 3.2 – Charge – Décharge avec écrouissage et contrainte résiduelle

La limite devient σ_e^1 et s'il y a recharge, le comportement redevient élastique selon la courbe définie par la figure 3.3 où $\sigma_e^1 > \sigma_e^0$: c'est l'écruissage :

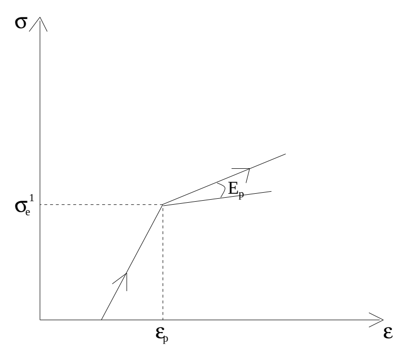

FIG. 3.3 – Comportement avec recharge

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3.2 Présentation du modèle

Même si le problème d'élastoplasticité avec écrouissage a été amplement abordé, à titre d'exemple [HKAM, LCLF, FJ, FL]. Le modèle ici présenté est distingué par une loi de comportement unifiée durant toutes les phases de chargement et de déchargement, d'élasticité

et de plasticité, ... et par un pas de calcul qui peut être grand.
Un modèle unifié d'élastoplasticité pouvant représenter les différents étages (charge, décharge, écrouissage,...) est défini par les étapes présentées ci-dessous et cela sous forme unidimensionnelle et sous sa forme tridimensionnelle (tensorielle). Le modèle permet automatiquement avec la même loi de comportement de :

1. Savoir si nous sommes dans la partie élastique ou plastique.
2. Modifier la contrainte limite suite à un écrouissage.

À chaque pas de chargement, un accroissement de déformation $d\epsilon$ initie l'étape. un calcul élastique est effectué, le modèle détecte avec la même procédure et en utilisant la fonction g définie par $g(\sigma) = \frac{1}{\pi}(\frac{\pi}{2} + arctg(\sigma))$, si le comportement est élastique ou bien plastique. Dans le cas d'une décharge, une nouvelle limite est définie : c'est l'écrouissage.

Forme unidimensionnelle	Forme tridimensionnelle
$\epsilon_0; \sigma_0; \bar{\sigma_c}; \bar{\sigma_T}$	$\epsilon^0_{ij}; \sigma^0_{ij}; \bar{\sigma_0}$
$d\epsilon$	$d\epsilon_{ij}$
$\epsilon = \epsilon_0 + d\epsilon$	$\epsilon_{ij} = \epsilon^0_{ij} + d\epsilon_{ij}$
$d\sigma_e = kd\epsilon$: k est la rigidité	$d\sigma^e_{ij} = k_{ijkl} d\epsilon_{lk}$
$\sigma_e = \sigma_0 - d\sigma_e$	$\sigma^e_{ij} = \sigma^0_{ij} - d\sigma^e_{ij}$
$\bar{\sigma} = \bar{\sigma_c}.g_c(\sigma_e) + \bar{\sigma_T}.g_T(\sigma_e)$	σ_{VM}
$\sigma_{00} = \sigma_0 + (\bar{\sigma} - \sigma_0).g_1(\lvert\frac{\sigma_e}{\bar{\sigma}}\rvert)$	$\sigma^{00}_{ij} = \sigma^0_{ij} + (\frac{\bar{\sigma}}{\sigma_{VM}}.\sigma^e_{ij} - \sigma^0_{ij}).g_1(\lvert\frac{\sigma_{VM}}{\bar{\sigma}}\rvert)$
$\epsilon_{00} = \epsilon_0 - \frac{(\bar{\sigma} - \sigma_0.g_1(\lvert\frac{\sigma_e}{\bar{\sigma}}\rvert)}{k}$	$\epsilon^{00}_{ij} = \epsilon^0_{ij} + k^{-1}_{ijkl}(\frac{\bar{\sigma}}{\sigma_{VM}}.\sigma^e_{ij} - \sigma^0_{ij}).g_1(\lvert\frac{\sigma_{VM}}{\bar{\sigma}}\rvert)$
$d\epsilon = \epsilon - \epsilon_{00}$: déformation	$d\epsilon_{ij} = \epsilon_{ij} - \epsilon^{00}_{ij}$
$d\sigma = 2k.\frac{(1 - g_1(\lvert\frac{\sigma_e}{\bar{\sigma}}\rvert))}{(2 - g_1(\lvert\frac{\sigma_e}{\bar{\sigma}}\rvert))} d\epsilon$	$d\sigma_{ij} = 2k_{ijkl}.\frac{(1 - g_1(\lvert\frac{\sigma_{VM}}{\bar{\sigma}}\rvert))}{(2 - g_1(\lvert\frac{\sigma_{vm}}{\bar{\sigma}}\rvert))} d\epsilon_{lk}$
$\sigma = \sigma_{00} + d\sigma$	$\sigma_{ij} = \sigma^{00}_{ij} + d\sigma_{ij}$
$\sigma_0 = \sigma$	$\sigma^0_{ij} = \sigma_{ij}$
$\bar{\sigma_c} = \bar{\sigma_c} + (\sigma - \bar{\sigma_c}).g_{1c}(\frac{\sigma_e}{\lvert\bar{\sigma}\rvert})$	σ_{VM}
$\bar{\sigma_T} = \bar{\sigma_T} - (\sigma - \bar{\sigma_T}).g_{1T}(\frac{\sigma_e}{\lvert\bar{\sigma}\rvert})$	$\bar{\sigma} = \bar{\sigma} + (\sigma_{VM} - \bar{\sigma}).g_1(\frac{\sigma_{VM}}{\lvert\bar{\sigma}\rvert})$

3.3 Résultats numériques

3.3.1 Exemple 1 : Contrainte uniaxiale de traction – compression

Soit un milieu ayant une contrainte uniaxiale de traction compression. Une évolution de la déformation a été faite en parcourant des états de déformations différentes : élastique, plastique, plastique avec écrouissage, changement de signe de la déformation, atteinte de la limiter en traction puis en compression (de signe différent), figure 3.4. Notons que l'algorithme explicitant le modèle permet d'avoir un pas de déformation assez important.

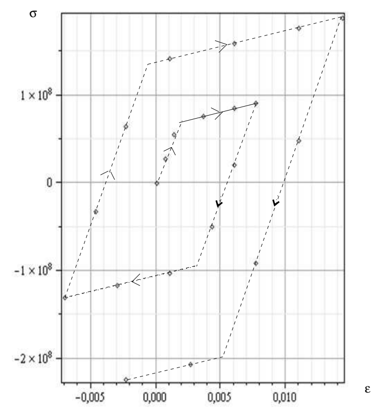

FIG. 3.4 – Contrainte – déformation pour le cas d'une contrainte unidimensionnelle

3.3.2 Exemple 2 : Etat de contrainte en cisaillement pure

Le deuxième exemple est un cisaillement pure en déformation plane. la structure est soumise à deux forces $\overrightarrow{T} et - \overrightarrow{T}$ perpendiculaires à l'axe où les points d'applications sont décalés.
Le tenseur de contrainte de cisaillement est sous la forme suivante :$\bar{\bar{\sigma}} = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}$

Si la contrainte de cisaillement τ est inférieure à la limite de cisaillement du matériau alors le comportement est élastique, si non il est plastique avec apparition d'écrouissage et nous observons le même phénomène qu'avec la traction, un nouveau seuil de plasticité à chaque augmentation de charge. figure 3.5

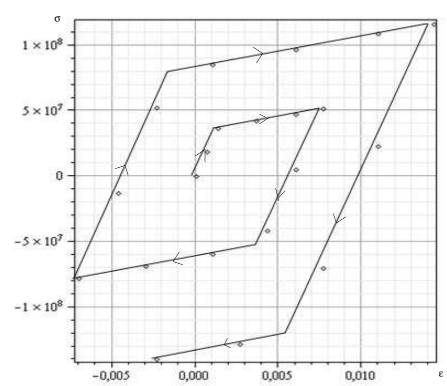

FIG. 3.5 – Cisaillement simple

Si nous appliquons une décharge très grande alors observons un résidu de déformation négatif comme l'indique l'exemple suivant la figure3.6

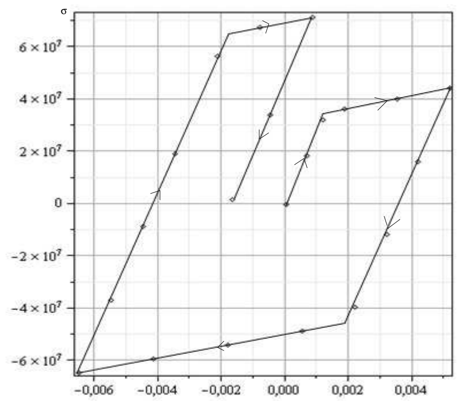

FIG. 3.6 – Cisaillement simple avec décharge importante

3.3.3 Exemple 3 : Etat de contrainte Tridimensionnelle

En tridimensionnelle, les résultats peuvent être plus intéressants, dans le sens ou les variations des déformations peuvent être variées. Dans ce cas la limite d'élasticité est définie par le critère de Von Mises défini par :

$$\sigma = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2)}$$

Le principe est le même qu'en unidimensionnelle. Si la contrainte de Von Misés est inférieure à la limite d'élasticité alors le comoprtement est élastique dans le cas contraire, il apparait un écrouissage.
Ce milieu est soumis à une évolution des déformations ϵ_{11}, ϵ_{22}, ϵ_{33} à valeurs égales (sans déformation de cisaillement) où son tenseur de contrainte est :

$\bar{\bar{\sigma}} = K\epsilon$ avec K matrice de rigidité et $\epsilon = \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \end{pmatrix}$

Les trois figures qui suivent figure 3.7, 3.8 et 3.9 représentent respectivement l'évolution des contraintes $\sigma_{11}, \sigma_{22}, \sigma_{33}$ en fonction des déformations respectives.
Les mêmes constatations qu'en unidimensionnelle ont été observées à chaque augmentation de charge, nous obtenons un nouveau seuil de plasticité avec apparition d'écrouissage.

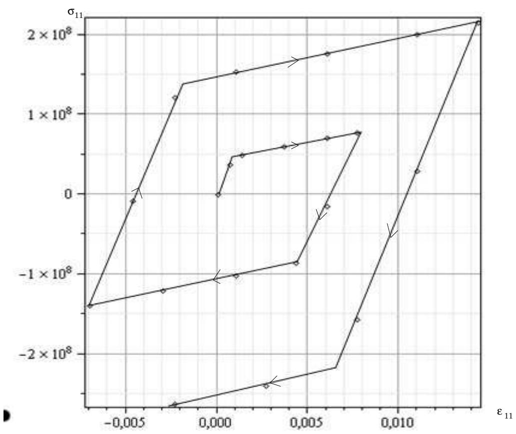

FIG. 3.7 – Évolution en déformation tridimensionnelle

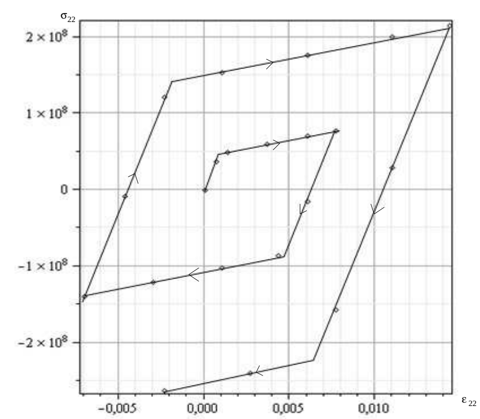

FIG. 3.8 – Évolution en déformation tridimensionnelle

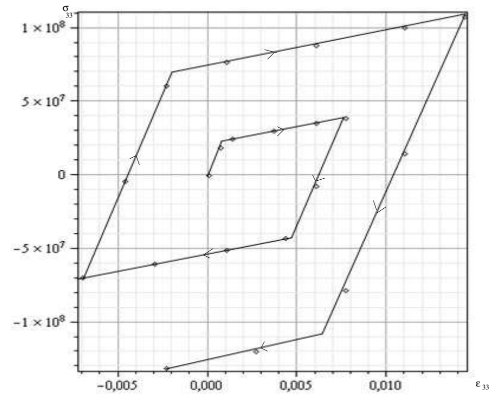

FIG. 3.9 – Évolution en déformation tridimensionnelle

3.4 Conclusion et perspectives

Aux deux chapitres traitant l'approche multiéchelle, le comportement du matériau avec défaut a été modélisé par une approche asymptotique cubique et quadratique. Le modèle présenté ici pouvait remplacer l'approche asymptotique et diminuer ainsi le temps de calcul. Un couplage est donc prévu avec l'approche multiéchelle.

CONCLUSION

La problématique de l'analyse des vibrations non linéaires des structures avec défauts localisés a été étudiée selon l'approche de développement asymptotique à double ou triple échelle ; la structure mécanique est auparavant discrétisée par éléments finis ce qui fournit un système d'équations différentielles non linéaires du deuxième ordre. Pour déterminer les solutions du système, la méthode du développement asymptotique à échelles multiples a été adoptée. L'intérêt pratique de ce développement vient justement du fait que l'on peut obtenir une bonne approximation dans une bande de fréquences pour des problèmes de vibration comportant un petit paramètre de perturbation qui produit une modulation lentement variable des vibrations. L'analyse mathématique de validité du développement asymptotique de solutions périodiques des vibrations libres (dites modes normaux non linéaires) a été rigoureusement faite et la détermination des modes propres d'un système non linéaire à quelques degrés de liberté a été analysé. Les résultats obtenus par cette approche ont été comparés avec une résolution numérique utilisant un programme d'intégration numérique pas à pas de scilab ; d'autre part, une transformation de Fourier numérique a été effectuée pour déterminer les fréquences. Celles ci ont été comparées avec celles du système linéaire. Les solutions de ce dernier peuvent tout au moins servir de références pour identifier les effets de non linéarité localisée sur le comportement non linéaire global.

L'analyse a été faite sur un système non linéaire dans la base des vecteurs propres pour simplifier les calculs en suivant le principe des modes normaux qui consiste à ce que toutes les composantes vibrent à la même fréquence. Des résultats illustrés par des exemples ont été exposés.

Pour les vibrations forcées amorties avec une fréquence excitatrice proche de la fréquence propre de vibration libre, la résolution est complexe, néanmoins, nous sommes arrivés à décrire un phénomène de résonance étendant la résonance classique du cas linéaire.

On a mis en évidence une solution périodique du système amorti et forcé ; elle apparaît comme solution stationnaire du système transformé ; la stabilité de la solution au voisinage de celle ci est importante en elle même mais est aussi un ingrédient de la démonstration de la convergence du développement asymptotique effectué. Nous avons montré que l'amplitude de la solution périodique atteint une valeur maximale pour une fréquence de la force appliquée proche de celle d'une vibration libre (*résonance primaire*). Nous avons remarqué dans l'expression de la fréquence angulaire de la solution périodique amortie et forcée, qu'aux termes trouvés dans le cas de vibration libre s'ajoute un terme impliquant le facteur d'amortissement. Pour cette fréquence, nous avons obtenu une solution approchée périodique à l'ordre ϵ. Avec cette approche nous mettons en évidence que lorsque la solution stationnaire atteint son amplitude maximum, la force excitatrice satisfait la relation suivante $F = \lambda \omega a_0^*$, ainsi, le terme d'amortissement λ peut être déterminé, c'est assez intéressant en pratique car il est en général difficile à mesurer. Par contre, Il a été observé que le terme quadratique n'a aucune influence sur la réponse approchée obtenue avec le développement à double échelle .

En utilisant le développement à triple échelle, l'expression de la solution approchée de la fréquence de la solution périodique est améliorée. Dans l'expression de la fréquence, aux termes trouvés avec deux échelles de temps s'ajoute un terme d'ordre ϵ^2 qui fait intervenir les coefficients des termes quadratiques et cubiques de l'équation du système. L'introduction d'une autre échelle de temps a compliqué la détermination de l'amplitude et de la phase. Pour palier à cette difficulté, nous avons eu recourt à la méthode de la reconstitution [Nay05] qui semble satisfaire la problématique mécanique ; nous avons donné une preuve mathématique de son utilisation. Un lien entre la fréquence de la solution libre et l'amplitude de la solution forcée a été mis en évidence. Il permet de déterminer une approximation de la fréquence de résonance du système forcé ; elle est trouvée proche de la fréquence du mode normal non linéaire.

Perspectives : les points suivants peuvent être considérés :

- Le décalage de la fréquence dépend de la position du défaut, ceci permet d'envisager un problème inverse de la localisation du défaut, par suite ; plus directement l'identification des paramètres de rigidité linéaire, ainsi que les coefficients des rigidités non linéaires c et d à partir de mesures expérimentales de la réponse du système peuvent être considérés.
- Une utilisation de cette approche dans le cas de plaques est une possibilité pratique très utile en ingéniérie.
- Associer le modèle élasto-plastique unifié et les approches à multiples échelles.
- Tester le modèle élasto-plastique unifié sur des exemples industriels avec un grand nombre de degrés de libertés.

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